

POISSON AND NEAR-SYMPLECTIC STRUCTURES ON GENERALIZED WRINKLED FIBRATIONS IN DIMENSION 6

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ABSTRACT. We show that generalized broken fibrations in arbitrary dimensions admit rank-2 Poisson structures compatible with the fibration structure. After extending the notion of wrinkled fibration to dimension 6 we prove that these wrinkled fibrations also admit compatible rank-2 Poisson structures. In the cases with indefinite singularities we can provide these wrinkled fibrations in dimension 6 with near-symplectic structures.

1. Introduction

Since the seminal work of Donaldson, establishing a correspondence between Lefschetz pencils and symplectic 4-manifolds [8], Lefschetz fibrations and its generalizations have played a significant role in symplectic geometry. These are maps to the 2-sphere with a finite number of isolated singular points where the rank of the derivative is zero. In 2005, Auroux, Donaldson, and Katzarkov generalized this approach, introducing what is now known as a *broken Lefschetz fibration* or *bLf* [1]. There is an additional component in the singularity set of bLfs, a 1-submanifold of indefinite folds. It was shown that a reciprocal geometric structure to bLfs is a near-symplectic form. The latter are closed 2-forms that are non-degenerate outside a collection of circles where they vanish, and they are known to exist on any 4-manifold with $b_2^+ > 0$. Recently, near-symplectic structures and generalized broken Lefschetz fibrations have been studied in higher dimensions [19].

From a singularity theory point of view bLfs are not stable. By a stable map it is understood one such that any nearby map in the space of smooth mappings can be perturbed to the original map after a change of coordinates in the domain and codomain. BLfs can be deformed to stable maps. Lekili showed that the unstable Lefschetz singularities of a bLf can be substituted by cusps, leading to a stable map with only folds and cusps as elements of its critical set [14]. These mappings are known as *wrinkled fibrations*. Furthermore he showed that the near-symplectic structure is preserved under these deformations.

Poisson geometry has newly entered the picture, particularly in dimension 4. A singular Poisson bivector of rank 2 that vanishes on the singularity set of a bLf and whose symplectic foliation matches the fibres of the map can be given [10]. This in turn implies that on any homotopy class of maps from a 4-manifold to S^2 there is such a singular Poisson structure. Similar structures also appear on wrinkled fibrations, where the local models of the Poisson bivectors and induced symplectic forms have been explicitly constructed [17].

In this work we start by defining a generalization of wrinkled fibrations on dimension 6 based on singularity theory. We then construct Poisson and near-symplectic structures that match the singularities of the fibration and give their local models. Before presenting these constructions, in section 2.1 we briefly recall the notion of a generalized broken Lefschetz fibration, which serves as a reference for the definition of generalized wrinkled fibrations. Our first observation appears

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after combining the definition of generalized bLf [19] together with the results of [6] and [10] on Poisson structures.

Theorem 1.1. *Let M and X be closed oriented smooth manifolds with $\dim(M) = 2n$, $\dim(X) = 2n - 2$, and $f: M \rightarrow X$ a generalized broken Lefschetz fibration. Then there is a complete singular Poisson structure of rank 2 whose associated bivector vanishes on the singularity set of f . If none of its symplectic leaves are, or contain, 2-spheres, then this Poisson structure is integrable.*

The proof of this theorem is a direct application of Theorem 2.11 and the definition of completeness. The integrability condition is verified in the relevant cases, as explained in 2.4.

In section 3 we focus on the Poisson structure on the total space of a generalized wrinkled fibration in dimension 6. We give the general steps for building Poisson bivectors around all types of singularities of corank 1, which can be applied on any given dimension. This idea allows us to show the following.

Theorem 1.2. *Let M be a closed, orientable, smooth 6-manifold equipped with a generalized wrinkled fibration $f: M \rightarrow X$ over a smooth 4-manifold X . Then there exists a complete Poisson structure whose symplectic leaves correspond to the fibres of the given fibration structure, and the singularities of both the fibration and the Poisson structures coincide. Moreover, for each singularity, the Poisson bivector and induced symplectic form on the leaves are given by the following equations:*

Folds:	Poisson bivectors (3.3), (3.6), (3.7)	and symplectic forms (4.5), (4.6), (4.7)
Cusps:	Poisson bivector (3.8), (3.11), (3.12)	and symplectic forms (4.8), (4.9)
Swallowtail:	Poisson bivector (3.13), (3.16), (3.17)	and symplectic forms (4.10), (4.11)
Butterfly:	Poisson bivector (3.18), (3.21), (3.22)	and symplectic forms (4.12), (4.13)

If none of its symplectic leaves are, or contain, 2-spheres, then this Poisson structure is integrable.

The existence of a Poisson structure with the stated properties follows from Theorem 2.11, previously shown by the first and third named authors together with García-Naranjo [10]. The proof of this theorem follows from an application of Theorem 2.11 and the definition of completeness.

These results allow us to present in section 2.5.1 countably many examples of Poisson structures on the same underlying smooth manifold that are Morita inequivalent. In our examples the leaves of the symplectic foliations change topology, as the fibrations involved undergo deformations.

In section 3, the local models for the bivectors are shown to hold true. Then in section 4 we prove that the local models for the symplectic forms on the leaves are also the claimed ones.

Finally, in section 5 we turn to near-symplectic geometry. Explicit models of near-symplectic forms have appeared in previous work [11, 1, 14, 19]. Here we show a further construction of local near-symplectic forms that follows a general scheme for all the singularities of a generalized wrinkled fibration. Assuming the global conditions for the existence of a near-symplectic structure on a $2n$ -manifold with a generalized bLf are met and constructing the local forms around the new singularities we obtain:

Theorem 1.3. *Let M be a closed oriented 6-manifold, (X, ω_X) a closed symplectic 4-manifold, and $f: M \rightarrow X$ a generalized wrinkled fibration. Denote by Z the singularity set of f , a 3-submanifold of M . Assume that there is a class $\alpha \in H^2(M)$, such that it pairs positively with every component of every fibre, and $\alpha|_Z = [\omega_X|_Z]$. Then there exist a near-symplectic form ω on M with singular locus Z such that it restricts to a symplectic form on the smooth fibres of the fibration.*

The proof of this theorem appears in section 5.2. The global construction of the near-symplectic structure on the total space follows as in dimension 4 [1, 14] with the modifications for higher dimensions introduced by the third named author [19]. The most substantial difference concerns the construction of the local forms around the new singularities.

Notice that our Theorem 1.2 presents extensive possibilities for Poisson structures supported on wrinkled fibrations. Whereas our Theorem 1.3 is limited in the singularities used by the restrictions imposed by near-symplectic structures. In particular, there will be no analogue of the results of Auroux-Donaldson-Katzarkov (for bLf's) or of Lekili (for wrinkled fibrations) in dimension 4, which give general conditions for all these fibrations to admit compatible near-symplectic structures.

Here we examine near-symplectic forms and singular Poisson bivectors in relation to extensions of Lefschetz fibrations using maps coming from singularity theory. The recent work of Cavalcanti and Klaasse [5] also considered Lefschetz fibrations in singular symplectic and Poisson geometry. In contrast to our work, Cavalcanti and Klaasse have connected achiral Lefschetz fibrations to log-symplectic and folded symplectic structures. An even dimensional manifold M is said to be *log-symplectic* if it is equipped with a Poisson bivector π such that the Pfaffian π^n is transverse to the zero section in $\Lambda^{2n}TM$, M is called *folded symplectic* if it comes with a closed 2-form ω such that ω^n vanishes transversally in $\Lambda^{2n}T^*M$. The geometric structures studied in this work differ from the ones considered by Cavalcanti and Klaasse. It is not yet clear what is the precise relationship between broken Lefschetz fibrations and log-symplectic manifolds.

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2. Preliminaries

2.1. Wrinkled and Broken Lefschetz Fibrations

We start by recalling the definition of a broken Lefschetz fibration. Regular fibres are 2-dimensional, smooth and convex and singular fibres present an isolated nodal singularity.

Definition 2.1. *On a smooth, closed 4-manifold X , a broken Lefschetz fibration or BLF is a smooth map $f: X \rightarrow S^2$ that is a submersion outside the singularity set. Moreover, the allowed singularities are of the following type:*

- (i) *indefinite fold singularities, also called broken, contained in the smooth embedded 1-dimensional submanifold $\Gamma \subset X \setminus C$, which are locally modelled by the real charts*

$$\mathbb{R}^4 \rightarrow \mathbb{R}^2, \quad (t, x_1, x_2, x_3) \mapsto (t, -x_1^2 + x_2^2 + x_3^2),$$

- (ii) *Lefschetz singularities: finitely many points*

$$C = \{p_1, \dots, p_k\} \subset X,$$

which are locally modeled by complex charts

$$\mathbb{C}^2 \rightarrow \mathbb{C}, \quad (z_1, z_2) \mapsto z_1^2 + z_2^2.$$

We recollect a few concepts of singularity theory before defining a generalized bLf. Let $f: M^n \rightarrow X^q$ be a smooth map between two smooth manifolds with $\dim(M) \geq \dim(X)$ and differential map $df: TM \rightarrow TX$. A point $p \in M$ is *regular* if the rank of df_p is maximal. In this case f is a submersion at p . If $\text{Rank}(df_p) < \dim(X)$, then a point $p \in M$ is called a *singularity* of f . Let $k = \dim(X) - \text{Rank}(df_p)$ denote the corank of f . The set $\Sigma_k = \{p \in M \mid \text{corank}(df_p) = k \geq 1\}$ is known as the *singularity set* or *singular locus* of f . For generic maps, Σ_k are submanifolds of M . As we can see from the definition, there can be different singularity sets depending on the corank of f . In this work we will focus on singularities of corank 1. The elements of the set Σ_1 satisfying $T_p\Sigma_f + \ker(f) = T_pM$ are called *fold singularities* of f . A mapping $f: M \rightarrow X$ is then known

as a *submersion with folds*, if it is a submersion outside the set of fold singularities. In particular, a submersion with folds restricts to an immersion on its fold locus (see Lemma 4.3 p.87 [12]). Submersions with folds are related to stable maps. By a stable f we mean that any nearby map $\tilde{f} \in C^\infty(M, X)$ is equivalent to f after a smooth change of coordinates in the domain and range.

Folds are locally modelled by real coordinate charts $\mathbb{R}^n \rightarrow \mathbb{R}^q$ with $n > q$ and coordinates

$$(x_1, \dots, x_n) \rightarrow (x_1, \dots, x_{q-1}, \pm x_q^2 \pm x_{q+1}^2 \pm \dots \pm x_n^2)$$

As we can see from the above parametrization, when $q = 1$, submersions with folds correspond precisely to Morse functions on M . It is well known that Morse functions are dense in the set of smooth mappings from any n -dimensional manifold M to \mathbb{R} . There is an equivalent statement for maps with a 2-dimensional target space. Assuming that f is generic then Σ_1 is a submanifold, and the restriction of f at Σ_1 gives a smooth map between manifolds that can also have generic singularities. When the target map is of dimension 2, there is one extra type of generic singularity called cusp. *Cusps* are points $p \in \Sigma_1$ such that $T_p \Sigma_1(f) = \ker(df_p)$, and they are parametrized by real charts $\mathbb{R}^n \rightarrow \mathbb{R}^2$ with coordinates

$$(x_1, \dots, x_n) \rightarrow (x_1, x_2^3 + x_1 \cdot x_2 \pm x_3^2 \pm \dots \pm x_n^2)$$

Folds and cusps are the singularities of a wrinkled fibration.

Definition 2.2. *A purely wrinkled fibration is a submersion f on a closed 4-manifold X to a closed surface having two types of singularities:*

- (i) *indefinite fold singularities contained in the smooth embedded 1-dimensional submanifold $\Gamma \subset X$, which are locally modelled by the real charts*

$$\mathbb{R}^4 \rightarrow \mathbb{R}^2, \quad (t, x_1, x_2, x_3) \mapsto (t, -x_1^2 + x_2^2 + x_3^2),$$

- (ii) *cusps, finitely many points contained in the set $B = \{p_1, \dots, p_k\} \subset X$, which are locally modeled by real charts*

$$\mathbb{R}^4 \rightarrow \mathbb{R}^2, \quad (t, x_1, x_2, x_3) \mapsto (t, x_1^3 + 3tx_1 + x_2^2 - x_3^2).$$

The following theorem shows that generic maps from any n -dimensional manifold to a 2-dimensional base have folds and cusps.

Theorem 2.3. [12] *A generic smooth map $M^n \rightarrow N^2$ has folds and cusps singularities.*

In this context, wrinkled fibrations are generic maps defined on a smooth 4-manifold with image on the 2-sphere, and broken Lefschetz fibrations are submersions with folds and Lefschetz singularities. The latter are natural in the symplectic setting. As it was shown by Donaldson and Gompf, there is a correspondence between symplectic 4-manifolds and Lefschetz fibrations. Yet, Lefschetz singularities are not stable from the point of view of singularity theory. Lekili showed that Lefschetz singularities can be transformed into cusps yielding to a wrinkled fibration. As a consequence, we can modify a bLf into a submersion with folds and cusps, which are stable and dense.

We proceed now to higher dimensions. To start, consider the definition of broken Lefschetz fibrations in higher dimensions [19].

Definition 2.4. *Let M, X be smooth manifolds of dimensions $2n$ and $2n - 2$. By a generalized broken Lefschetz fibration we mean a submersion $f: M \rightarrow X$ with two types of singularities:*

1. *Indefinite fold singularities, locally modeled by:*

$$\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n-2}$$

$$(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, -x_1^2 + x_2^2 + x_3^2)$$

The fold locus is an embedded codimension 3 submanifold. We denote it by Z . Singular fibres have again at most one singularity on each fibre, but this time crossing Z changes the genus of the regular fibre by one. Throughout this work we assume that the singular fibres do not intersect each other.

2. Lefschetz-type singularities, locally modelled by:

$$\begin{aligned}\mathbb{C}^n &\rightarrow \mathbb{C}^{n-1} \\ (z_1, \dots, z_n) &\rightarrow (z_1, \dots, z_{n-2}, z_{n-1}^2 + z_n^2)\end{aligned}$$

These singularities are contained in codimension 4 submanifolds cross a Lefschetz singular point. We denote the set of Lefschetz-type singularities by C . Each singular fibre presents at most one singularity on each fibre. On a piece of the fibre, this can be depicted as a local cone that collapses at the origin where $z_{n-1}^2 + z_n^2 = 0$. Nearby fibres are smooth. In the local description on a piece of a fibre, the cone opens up again and it is convex.

2.2. Generalized Wrinkled Fibrations

Stable maps of $M^n \rightarrow X^q$ are dense in $C^\infty(M, X)$ if and only if the pair (n, q) satisfies certain conditions depending on the dimension q of the target manifold X and the difference $(n - q)$. We refer the reader to [12, 15] for a detailed account. In particular, in the case of $M^6 \rightarrow X^4$ we have the following characterization.

Theorem 2.5. [15, 12] *A generic smooth map $M^6 \rightarrow N^4$ has folds, cusps, swallowtails, and butterflies singularities.*

This suggests the following definition.

Definition 2.6. *On a smooth 6-manifold M a generalized wrinkled fibration $f: M \rightarrow X$ is a submersion to a smooth closed 4-manifold X with the following four indefinite singularities each locally modelled by real charts $\mathbb{R}^6 \rightarrow \mathbb{R}^4$*

1. folds

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, -x_1^2 + x_2^2 + x_3^2)$$

2. cusps

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1 \cdot x_1 + x_2^2 - x_3^2)$$

3. swallowtails

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1 x_1^2 + t_2 x_1 + x_2^2 - x_3^2)$$

4. butterflies

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1 x_1^3 + t_2 x_1^2 + t_3 x_1 + x_2^2 - x_3^2)$$

2.3. Poisson Manifolds

This section contains basic facts about Poisson geometry that we will use throughout the paper. We refer the interested readers to [18, 9, 13] for further details.

2.3.1. Poisson Structures

Definition 2.7. *A Poisson bracket (or a Poisson structure) on a smooth manifold M is a bilinear operation $\{\cdot, \cdot\}$ on the set $C^\infty(M)$ of real valued smooth functions on M that satisfies*

- (i) $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra.
- (ii) $\{gh, k\} = g\{h, k\} + h\{g, k\}$ for any $g, h, k \in C^\infty(M)$.

A manifold M with such a Poisson bracket is called a *Poisson manifold*. If (M, ω) is a symplectic manifold, we may use the symplectic form ω to produce a Poisson structure. The bracket of M is defined by

$$\{g, h\} = \omega(X_g, X_h).$$

Hamiltonian vector fields X_h are defined by $\mathbf{i}_{X_h}\omega = dh$.

Thus, using property (ii) from Definition 2.7 we may define Hamiltonian vector fields for Poisson manifolds. Given a function $h \in C^\infty(M)$ we assign it the *Hamiltonian vector field* X_h , defined via

$$X_h(\cdot) = \{\cdot, h\}.$$

It follows from (ii) that a Poisson bracket $\{g, h\}$ depends only on the first derivatives of g and h . Hence the Poisson bracket may be considered as defining a bivector field π defined by

$$(2.1) \quad \{g, h\} = \pi(dg, dh).$$

Let π be a Poisson bivector, for coordinates (x^1, \dots, x^n) , we give a local expression

$$\pi(x) = \frac{1}{2} \sum_{i,j=1}^n \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

Here $\pi^{ij}(x) = \{x^i, x^j\} = -\{x^j, x^i\}$.

A Poisson bracket satisfies the Jacobi identity, this is expressed by a partial differential equation for the components of the Poisson bivector [13]. Maps that preserve Poisson brackets are called Poisson maps. There are also maps which are called *anti-Poisson*:

Definition 2.8 (anti-Poisson maps). *A map $\varphi : P \rightarrow P'$ between two Poisson manifolds P and P' is called an anti-Poisson map if*

$$\{f_1 \circ \varphi, f_2 \circ \varphi\}_{P'} = -\{f_1, f_2\} \circ \varphi$$

holds for any two smooth functions f_1 and f_2 on P .

Let π be a bivector on M , $q \in M$, and $\alpha_q \in T_q^*M$. It is possible to define a bundle map:

$$\mathcal{B} : T^*M \rightarrow TM ; \mathcal{B}_q(\alpha_q)(\cdot) = \pi_q(\cdot, \alpha_q)$$

If π is a Poisson bivector, we obtain that $X_h = \mathcal{B}(dh)$. We then define the *rank* of π at $q \in M$ to be equal to the rank of $\mathcal{B}_q : T_q^*M \rightarrow T_qM$. This coincides with the rank of the matrix $\pi^{ij}(x)$.

The distribution defined by \mathcal{B}_q on T_qM is called the *characteristic distribution* of π . We know by the *Symplectic Stratification Theorem* that the characteristic distribution of a Poisson bivector π gives rise to a (possibly singular) foliation by symplectic leaves. This foliation is integrable in the sense of Stefan-Sussman (see Theorem 2.6 in [18]).

Call Σ_q the symplectic leaf of M through the point q . As a set Σ_q can be also considered as the collection of points that may be joined via piecewise smooth integral curves of Hamiltonian vector fields. Note that if ω_{Σ_q} is the symplectic form on Σ_q , $T_q\Sigma_q$ is exactly the characteristic distribution of π through p . Therefore, given $u_q, v_q \in T_q\Sigma_q$ there exist $\alpha_q, \beta_q \in T_q^*M$ such that its image \mathcal{B}_q is u_q and v_q , respectively. Using this we can describe the symplectic form ω_{Σ_q} :

$$(2.2) \quad \omega_{\Sigma_q}(q)(u_q, v_q) = \pi_q(\alpha_q, \beta_q) = \langle \alpha_q, v_q \rangle = -\langle \beta_q, u_q \rangle.$$

The rank varies as the dimensions of the symplectic leaves do. When the rank of the characteristic distribution of a bivector is less than or equal to two the following holds [10]:

Proposition 2.9.

- (i) *If π is a bivector field on M whose characteristic distribution is integrable and has rank less than or equal to two at each point, then π is Poisson.*

- (ii) Let π be a Poisson structure on M whose rank at each point is less than or equal to two. Then $\pi_1 := k\pi$ is also a Poisson structure where $k \in C^\infty(M)$ is an arbitrary non-vanishing function.

We will describe the bivectors locally using certain Casimir functions.

Definition 2.10. Let M be a Poisson manifold. A function $h \in C^\infty(M)$ is called a Casimir if $\{h, g\} = 0$ for every $g \in C^\infty(M)$. Equivalently $\mathcal{B}(dh) = 0$.

Theorem 2.11 ([10]). Let M be an orientable n -manifold, N an orientable $n - 2$ manifold, and $f : M \rightarrow N$ a smooth map. Let μ and Ω be orientations of M and N respectively. The bracket on M defined by

$$(2.3) \quad \{g, h\}_\mu = k dg \wedge dh \wedge f^* \Omega$$

where k is any non-vanishing function on M is Poisson. Moreover, its symplectic leaves are

- (i) the 2-dimensional leaves $f^{-1}(s)$ where $s \in N$ is a regular value of f ,
- (ii) the 2-dimensional leaves $f^{-1}(s) \setminus \{\text{Critical Points of } f\}$ where $s \in N$ is a singular value of f .
- (iii) the 0-dimensional leaves corresponding to each critical point.

Formula (2.3) appeared in [6] (attributed to H. Flaschka and T. Ratiu).

2.3.2. Completeness

We recall the concept of a complete Poisson manifold

Definition 2.12. A Poisson manifold M is said to be complete if every Hamiltonian vector field on M is complete.

Notice that M is complete if and only if every symplectic leaf is bounded in the sense that its closure is compact. We can also talk about complete Poisson maps.

Definition 2.13. A Poisson map $J : Q \rightarrow P$ between the Poisson manifolds Q and P is complete if the hamiltonian vector field X_{J^*f} is complete whenever X_f is complete, f in $C^\infty(P)$.

2.4. Integrability

2.4.1. Lie Algebroids

A **Lie algebroid** $(A, B, \rho, [\cdot, \cdot])$ is a vector bundle A over a manifold B together with a bundle map $\rho : A \rightarrow TB$, called the **anchor**, and a Lie bracket $[\cdot, \cdot]$ on the real vector space $\Gamma(A)$ of sections of A such that for every X, Y in $\Gamma(A)$ and smooth real function f of B :

$$[X, fY] = f[X, Y] + L_{\rho_* X} f X$$

Here ρ_* is the induced map on sections, and L the Lie derivative.

2.4.2. Lie Groupoid

A **groupoid** is small category where all the morphisms are invertible. This consists of a set of morphisms G and a set of objects B . There exist surjective maps $s, t : G \rightarrow B$ called the **source** and **target** maps, respectively. Let

$$G^{(2)} := \{(u, v) \in G \times G : s(u) = t(v)\}.$$

There exists a **multiplication** map;

$$G^{(2)} \rightarrow G; (u, v) \mapsto uv$$

an **inverse** map;

$$G \rightarrow G; u, v \mapsto u \mapsto (u^{-1})$$

and an **identity** bisection:

$$\epsilon : B \rightarrow G$$

These comply with the following axioms, for every u, v, w in G , and b in B .

$$\begin{aligned} s(uv) &= s(v), & t(uv) &= t(u), & \epsilon(b)v &= v, \\ u\epsilon(b) &= u, & (uv)w &= u(vw), & s(u^{-1}) &= t(u), \\ t(u^{-1}) &= s(u), & uu^{-1} &= \epsilon(t(u)), & uu^{-1} &= \epsilon(s(u)). \end{aligned}$$

The vector bundle $\ker(ds)|_{\epsilon(B)}$ has a natural structure of a Lie algebroid over B with anchor dt , its Lie bracket is induced by the multiplication.

Denote by $\text{Lie}(G)$ the Lie algebroid of the Lie groupoid G . Notice that not every Lie algebroid occurs as $\text{Lie}(G)$ for some G . Those which do are called **integrable**.

2.4.3. Algebroids on cotangent bundles to Poisson manifolds

The cotangent bundle of a Poisson manifold M can be given a Lie algebroid structure. The Poisson bivector field induces an anchor map, for every x in M and σ in T_x^*M :

$$\pi^\#(x)(\sigma) = \pi(x)(\sigma, \cdot)$$

The Lie bracket in this structure was introduced by Koszul for 1-forms:

$$[df, dg] := d\{f, g\}$$

When this Lie algebroid is integrable, its associated Poisson manifold is said to be integrable.

Crainic and Fernandes found general obstructions for the integrability of Lie algebroids. They proved that Poisson manifolds whose symplectic leaves have trivial second homotopy groups are integrable [3].

Definition 2.14. *A symplectic realization of a Poisson manifold P is a Poisson map from a symplectic manifold to P .*

2.5. Morita equivalences

We will now briefly recall the notion of Morita equivalence for integrable Poisson manifolds (see [2, 22, 23]).

Definition 2.15 (complete full dual pair). *Let S be a symplectic manifold. A pair of Poisson maps $P_1 \leftarrow S \rightarrow P_2$ is called a dual pair if the J_1 - and J_2 -fibres are the symplectic orthogonal of each other. Such a pair is called full if J_1 and J_2 are surjective submersions. If both J_1 and J_2 are complete, it is called complete.*

Definition 2.16 (Morita equivalence for integrable Poisson manifolds). *A pair of integrable Poisson manifolds P_1 and P_2 are called Morita equivalent if there exists a symplectic manifold S with a complete Poisson map $J_1 : S \rightarrow P_1$ and a complete anti-Poisson map $J_2 : S \rightarrow P_2$ so that $P_1 \leftarrow S \rightarrow \overline{P_2}$ is a complete full dual pair for which the J_1 - and J_2 -fibres are simply connected.*

For the readers' convenience, we include the next well known statement:

Lemma 2.17 (Morita equivalences and fundamental groups of leaves). *For Morita equivalent Poisson manifolds, corresponding symplectic leaves have isomorphic fundamental groups.*

Proof. Let P_1 and P_2 be two Morita equivalent Poisson manifolds and $P_1 \leftarrow S \rightarrow \overline{P_2}$ be the associated complete full dual pair. Consider symplectic leaves L_1 in P_1 and L_2 in P_2 such that $N = J_1^{-1}(L_1) = J_2^{-1}(L_2)$. Then J_i maps N onto L_i , and the maps have simply connected fibres. So there exists an induced isomorphism of fundamental groups; $\pi_1(L_1) \cong \pi_1(N) \cong \pi_1(L_2)$. \square

2.5.1. Morita inequivalent structures

Example 2.18. *The near-symplectic cobordisms described by Perutz [16] for 4-dimensional manifolds can be used to describe examples of Morita inequivalent Poisson structures. For example, suppose there is a near-symplectic cobordism where the number of connected components of the critical set in the base changes, then so does the topology of the fibres in the respective fibrations in the start and end of the cobordism. Assume that none of the fibres in these fibrations were or contained 2-spheres in the initial part of the cobordism, and that the genus of the fibration increases along the cobordism. Then, by lemma 2.17, the associated Poisson structures on the boundaries of the cobordism are not Morita equivalent.*

Example 2.19. *The deformations of wrinkled fibrations introduced by Lekili [14] have been used to describe Poisson structures on the associated fibrations [17]. In a similar way to the previous example assume that M_0 is a closed smooth oriented 4-manifold with a wrinkled fibration whose fibres do not contain 2-spheres. Then the associated Poisson structure Π_0 is integrable [17]. Perform one of Lekili's deformations on (M_0, Π_0) which increases the fibre genus, then the resulting manifold (M_1, Π_1) is Poisson [17]. Then Lemma 2.17 implies (M_0, Π_0) and (M_1, Π_1) are not Morita equivalent. Iterating this process exhibits a countable abundance of Morita inequivalent structures on the same underlying smooth 4-manifold.*

3. Local Poisson bivectors for the proof of Theorem 1.2 .

We will now give explicit local descriptions for the Poisson structures and the corresponding symplectic forms in a neighbourhood singularities of generalized wrinkled fibrations in dimension 6. All of the expressions that we will give depend abstractly on an arbitrary choice of a non-vanishing function k in $C^\infty(M)$. See Proposition 2.9. Before proceeding we will describe the general strategy employed to find the local bivectors.

Step 1: Consider the coordinate functions C_1, C_2, C_3, C_4 that describe each fibration as Casimir functions for the Poisson structure that we want to find.

Step 2: Calculate the differentials $dC_i, i = 1, 2, 3, 4$.

Step 3: We use formula 2.3 to compute the skew-symmetric matrix with entries:

$$\pi^{ij} = \{x^i, x^j\}_\mu = dx^i \wedge dx^j \wedge dC_1 \wedge dC_2 \wedge dC_3 \wedge dC_4.$$

This matrix will then annihilate $dC_i, i = 1, 2, 3, 4$. It will give the endomorphism \mathcal{B} associated to a Poisson structure with $dC_i, i = 1, 2, 3, 4$, as Casimirs. The components of the bivector field will be given by:

$$\{x^i, x^j\} = \det(\epsilon^i, \epsilon^j, dC_1, dC_2, dC_3, dC_4)$$

Here ϵ^i is the 6×1 canonical basis column vector, whose i -th component is 1 and all others are zero.

Step 4: We then write the Poisson bivector using the skew-symmetric matrix entries.

3.1. General criterion for constructing Poisson bivectors on singularities

We extend the previous strategy to manifolds of dimension $2n$ when we have a singular submersion with singularities of corank 1. The following construction will describe a procedure that can be used to compute local expressions of Poisson structures and their corresponding symplectic forms. We will implement this scheme to study the 6-dimensional case. An explicit computation of the local models in dimension 6 appears in appendix A.1.

Proposition 3.1. Let q be a point that either has complex coordinates $q = (z_1, z_2, \dots, z_n)$ or real coordinates $(t_1, t_2, \dots, t_{2n-4}, t_{2n-3}, x_1, x_2, x_3)$. Let f be a smooth map given as either $f: \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ or $f: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n-2}$ such that

$$f(q) = (z_1, \dots, z_{n-2}, f_o(z_{n-1}, z_n))$$

or

$$f(q) = (t_1, t_2, \dots, t_{2n-4}, t_{2n-3}, f_o(t_1, \dots, t_{2n-3}, x_1, x_2, x_3)),$$

respectively. Here f_o is a smooth map which depends only on the last coordinates z_{n-1}, z_n or x_1, x_2, x_3 . Then we can produce a Poisson structure associated to the local model given by f . The Poisson bivector has the form:

$$\pi = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & \pi^{11} & \pi^{12} & \pi^{13} & \pi^{14} \\ 0 & \cdots & 0 & \pi^{21} & \pi^{22} & \pi^{23} & \pi^{24} \\ 0 & \cdots & 0 & \pi^{31} & \pi^{32} & \pi^{33} & \pi^{34} \\ 0 & \cdots & 0 & \pi^{41} & \pi^{42} & \pi^{43} & \pi^{44} \end{pmatrix}$$

where π^{ij} is the Poisson bivector of the map f_o . Then $\pi^{ii} = 0$ and $\pi^{ij} = \pi^{ji}$. Therefore the Poisson bivector has the local form:

$$\pi(x) = \sum_{i,j=1}^4 \left[\pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \right]$$

Proof. In the case when f is a complex map we use the real and imaginary parts of each coordinate function as a Casimir function for the Poisson structure that we want to find. That is, we will have $2n - 2$ Casimir functions:

$$\begin{aligned} C_i &= \operatorname{Re}(z_i) & 1 \leq i \leq n-2 \\ C_{i+n-2} &= \operatorname{Im}(z_i) & 1 \leq i \leq n-2 \\ C_{2n-3} &= \operatorname{Re}(f_o(z_{n-1}, z_n)) \\ C_{2n-2} &= \operatorname{Im}(f_o(z_{n-1}, z_n)) \end{aligned}$$

Now we compute the differential matrix of the map. It gives a matrix with a 2×4 -block corresponding to the derivatives of the real and complex part of f_o and ones on the principal diagonal.

$$(3.1) \quad D = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial t_{2n-3}} & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial x_1} & \frac{\partial C_{2n-2}}{\partial x_1} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial x_2} & \frac{\partial C_{2n-2}}{\partial x_2} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial x_3} & \frac{\partial C_{2n-2}}{\partial x_3} \end{pmatrix}$$

According to the formula (2.3) the coefficients of the bivector matrix are given by

$$\pi^{ij} = \text{Det} \left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \epsilon_i^1 & \epsilon_j^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \epsilon_i^{2n-4} & \epsilon_j^{2n-4} \\ 0 & \dots & 0 & \frac{\partial C_{2n-3}}{\partial t_{2n-3}} & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} & \epsilon_i^{2n-3} & \epsilon_j^{2n-3} \\ 0 & \dots & 0 & \frac{\partial C_{2n-3}}{\partial x_1} & \frac{\partial C_{2n-2}}{\partial x_1} & \epsilon_i^{2n-2} & \epsilon_j^{2n-2} \\ 0 & \dots & 0 & \frac{\partial C_{2n-3}}{\partial x_2} & \frac{\partial C_{2n-2}}{\partial x_2} & \epsilon_i^{2n-1} & \epsilon_j^{2n-1} \\ 0 & \dots & 0 & \frac{\partial C_{2n-3}}{\partial x_3} & \frac{\partial C_{2n-2}}{\partial x_3} & \epsilon_i^{2n} & \epsilon_j^{2n} \end{pmatrix} \right]$$

where ϵ_i^k and ϵ_j^k are canonical basis column vectors, whose i -th and j -th component, respectively is 1 and all others are zero. Note that it contains a identity matrix of dimension $(2n-4) \times (2n-4)$. Therefore the determinant is the same as of the following matrix

$$\begin{pmatrix} 0 & 0 & \epsilon_i^{2n-4} & \epsilon_j^{2n-4} \\ \frac{\partial C_{2n-3}}{\partial t_{2n-3}} & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} & \epsilon_i^{2n-3} & \epsilon_j^{2n-3} \\ \frac{\partial C_{2n-3}}{\partial x_1} & \frac{\partial C_{2n-2}}{\partial x_1} & \epsilon_i^{2n-2} & \epsilon_j^{2n-2} \\ \frac{\partial C_{2n-3}}{\partial x_2} & \frac{\partial C_{2n-2}}{\partial x_2} & \epsilon_i^{2n-1} & \epsilon_j^{2n-1} \\ \frac{\partial C_{2n-3}}{\partial x_3} & \frac{\partial C_{2n-2}}{\partial x_3} & \epsilon_i^{2n} & \epsilon_j^{2n} \end{pmatrix}$$

which gives the coordinates of the Poisson bivector associated to f_o .

When f is a real map, we take the coordinates functions as Casimir functions:

$$\begin{aligned} C_i &= t_i \quad 1 \leq i \leq 2n-3 \\ C_{2n-2} &= f_o(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \end{aligned}$$

The differential matrix of the map is

$$(3.2) \quad \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\partial C_{2n-2}}{\partial t_1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \frac{\partial C_{2n-2}}{\partial t_{2n-4}} \\ 0 & \dots & 0 & 1 & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} \\ 0 & \dots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_1} \\ 0 & \dots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_2} \\ 0 & \dots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_3} \end{pmatrix}$$

Then, the coefficients of the corresponding bivector matrix are given by

$$\pi^{ij} = \text{Det} \left[\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\partial C_{2n-2}}{\partial t_1} & \epsilon_i^1 & \epsilon_j^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \frac{\partial C_{2n-2}}{\partial t_{2n-4}} & \epsilon_i^{2n-4} & \epsilon_j^{2n-4} \\ 0 & \dots & 0 & 1 & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} & \epsilon_i^{2n-3} & \epsilon_j^{2n-3} \\ 0 & \dots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_1} & \epsilon_i^{2n-2} & \epsilon_j^{2n-2} \\ 0 & \dots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_2} & \epsilon_i^{2n-1} & \epsilon_j^{2n-1} \\ 0 & \dots & 0 & 0 & \frac{\partial C_{2n-2}}{\partial x_3} & \epsilon_i^{2n} & \epsilon_j^{2n} \end{pmatrix} \right]$$

We note that $\pi^{ij} = 0$ for $1 \leq i \leq 2n-4$ and $1 \leq j \leq 2n-4$. The rest of the coefficients can be computed with the following

$$Det \left[\begin{pmatrix} 1 & 0 & \epsilon_i^{2n-3} & \epsilon_j^{2n-3} \\ 0 & \frac{\partial C_{2n-2}}{\partial x_1} & \epsilon_i^{2n-2} & \epsilon_j^{2n-2} \\ 0 & \frac{\partial C_{2n-2}}{\partial x_2} & \epsilon_i^{2n-1} & \epsilon_j^{2n-1} \\ 0 & \frac{\partial C_{2n-2}}{\partial x_3} & \epsilon_i^{2n} & \epsilon_j^{2n} \end{pmatrix} \right]$$

In fact, the only nonzero coefficients are:

$$\begin{aligned} \pi^{23} &= \frac{\partial C_{2n-2}}{\partial x_3} \\ \pi^{24} &= \frac{\partial C_{2n-2}}{\partial x_2} \\ \pi^{34} &= \frac{\partial C_{2n-2}}{\partial x_3} \end{aligned}$$

The result follows. □

3.2. Poisson structures on generalized wrinkled fibrations in dimension 6.

We apply the general criterion presented above to the case of wrinkled fibrations on 6-manifolds. Let $q \in M$ be a point, and $k : M \rightarrow X, k(t_1, t_2, t_3, x_1, x_2, x_3)$, be a non-vanishing smooth function.

3.2.1. Poisson bivector near a fold singularity.

Indefinite fold

The local coordinate model around a fold singularity is given by the map:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, -x_1^2 + x_2^2 + x_3^2)$$

The resulting Poisson structure of a fold singularity is given by:

$$(3.3) \quad \pi = k \left[2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} - 2x_1 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

Definite fold

In addition, we also compute the Poisson bivector for definite singularities for each wrinkled fibration. In this case, they are locally modeled by (3.4) and (3.5):

$$(3.4) \quad (t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^2 + x_2^2 + x_3^2)$$

$$(3.5) \quad (t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^2 + x_2^2 + x_3^2)$$

Following the general computations as above, the Poisson bivectors are, respectively:

$$(3.6) \quad \pi = k \left[2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

and

$$(3.7) \quad \pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

3.2.2. Poisson bivector near a cusp singularity.

Indefinite cusp

The local coordinate model around a cusp singularity is given by:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1x_1 + x_2^2 - x_3^2)$$

The Poisson bivector in the local coordinates of a cusp singularity is given by:

$$(3.8) \quad \pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 3(x_1^2 - t_1) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

Definite cusp

For definite singularities in cusps, we obtain in each case (3.9) and (3.10):

$$(3.9) \quad (t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1x_1 + x_2^2 + x_3^2)$$

$$(3.10) \quad (t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1x_1 - x_2^2 - x_3^2)$$

The corresponding bivectors are, respectively:

$$(3.11) \quad \pi = k \left[2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 3(x_1^2 - t_1) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

and

$$(3.12) \quad \pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 3(x_1^2 - t_1) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

3.2.3. Poisson bivector near a swallowtail singularity.

Indefinite swallowtail

The local coordinate model around a swallowtail singularity is given by the map:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1x_1^2 + t_2x_1 + x_2^2 - x_3^2)$$

The Poisson bivector in the local coordinates of a swallowtail singularity is described by:

$$(3.13) \quad \pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (4x_1^3 + 2t_1x_1 + t_2) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

Definite swallowtail

For definite singularities:

$$(3.14) \quad (t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1x_1^2 + t_2x_1 + x_2^2 + x_3^2)$$

$$(3.15) \quad (t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1x_1^2 + t_2x_1 - x_2^2 - x_3^2)$$

The corresponding bivectors are, respectively:

$$(3.16) \quad \pi = k \left[2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (4x_1^3 + 2t_1x_1 + t_2) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

and

$$(3.17) \quad \pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (4x_1^3 + 2t_1x_1 + t_2) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

3.2.4. Poisson bivector near a butterfly singularity.

Indefinite butterfly

The local coordinate model around a butterfly singularity is given by:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1 x_1^3 + t_2 x_1^2 + t_3 x_1 + x_2^2 - x_3^2)$$

The Poisson bivector in the local coordinates of a butterfly singularity is described by:

$$(3.18) \quad \pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (5x_1^4 + 3t_1 x_1^2 + 2t_2 x_1 + t_3) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

Definite butterfly The singularity is modeled by the coordinates:

$$(3.19) \quad (t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1 x_1^3 + t_2 x_1^2 + t_3 x_1 + x_2^2 + x_3^2)$$

$$(3.20) \quad (t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1 x_1^3 + t_2 x_1^2 + t_3 x_1 - x_2^2 - x_3^2)$$

The corresponding bivectors are, respectively:

$$(3.21) \quad \pi = k \left[2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (5x_1^4 + 3t_1 x_1^2 + 2t_2 x_1 + t_3) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

and

$$(3.22) \quad \pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (5x_1^4 + 3t_1 x_1^2 + 2t_2 x_1 + t_3) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

3.3. Poisson bivectors on higher dimensional type $2n$ generalized wrinkled fibrations.

Lekili defined 4 moves, these include all the possible 1-parameter deformations of broken and wrinkled fibrations up to homotopy (see [14]). Lekili showed that any 1-parameter family deformation of a purely wrinkled fibration is homotopic (relative endpoints) to one which realises a sequence of births, merges, flips, their inverses, and isotopies staying within the class of purely wrinkled fibrations. For higher dimensions, we will introduce a generalized form of these deformations. We will use them to give local expressions for the associated Poisson bivectors and symplectic forms near singularities described by the deformations.

Consider the following maps $\mathbb{R} \times \mathbb{R}^{2n-1} \rightarrow \mathbb{R}^{2n-2}$, given by the equations below, and each depending on a real parameter s :

$$(3.23) \quad b_s(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) = (t_1, \dots, t_{2n-3}, x_1^3 - 3x_1(t_{2n-3}^2 - s) + x_2^2 - x_3^2)$$

$$(3.24) \quad m_s(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) = (t_1, \dots, t_{2n-3}, x_1^3 - 3x_1(s - t_{2n-3}^2) + x_2^2 - x_3^2)$$

$$(3.25) \quad f_s(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) = (t_1, \dots, t_{2n-3}, x_1^4 - x_1^2 s + x_1 t_{2n-3} + x_2^2 - x_3^2)$$

$$(3.26) \quad w_s(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) = (t_1, \dots, t_{2n-4}, t_{2n-3}^2 - x_1^2 + x_2^2 - x_3^2 + s t_{2n-3}, 2t_{2n-3} x_1 + 2x_2 x_3)$$

We will also need a generalized wrinkled fibration for dimensions greater than 6.

Definition 3.2. Let M be a smooth $2n$ -manifold, and X be a smooth closed $2n - 2$ -manifold. A type $2n$ -wrinkled fibration is a smooth map $f : M \rightarrow X$ that is a submersion with the following four indefinite singularities each locally modelled by real charts $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n-2}$

(i) folds

$$(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, -x_1 + x_2^2 + x_3^2)$$

(ii) *cusps*

$$(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, x_1^3 - 3t_1 \cdot x_1 + x_2^2 - x_3^2)$$

(iii) *swallowtails*

$$(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, x_1^4 + t_1 x_1^2 + t_2 x_1 + x_2^2 - x_3^2)$$

(iv) *butterflies*

$$(t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, x_1^5 + t_1 x_1^3 + t_2 x_1^2 + t_3 x_1 + x_2^2 - x_3^2)$$

Corollary 3.3. *For a non-vanishing smooth function k in $C^\infty(M)$ we have the following consequences:*

(1) *Let X be a closed smooth oriented and connected $2n$ -manifold, and $f : M \rightarrow X$ a generalized broken Lefschetz fibration. The Poisson structures in a neighborhood of the two type of singularities can be computed to obtain Poisson bivectors near the following singularities*

Lefschetz-type singularity

$$\begin{aligned} \pi = k & \left[(x_2^2 + x_3^2) \frac{\partial}{\partial t_{2n-3}} \wedge \frac{\partial}{\partial x_1} + (x_1 x_2 - t_{2n-3} x_3) \frac{\partial}{\partial t_{2n-3}} \wedge \frac{\partial}{\partial x_2} - (t_{2n-3} x_2 + x_1 x_3) \frac{\partial}{\partial t_{2n-3}} \wedge \frac{\partial}{\partial x_3} \right. \\ & \left. + (t_{2n-3} x_2 + x_1 x_3) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + (x_1 x_2 - t_{2n-3} x_3) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + (t_{2n-3}^2 + x_1^2) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right] \end{aligned}$$

Indefinite fold singularity

$$\pi = k \left[x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \right]$$

(2) *Let M be a closed, orientable, smooth $2n$ -manifold endowed with a type $2n$ -wrinkled fibration f to a closed $2n - 2$ manifold X . Then a complete Poisson structure is given by the following bivectors near the corresponding singularities:*

Fold

$$\pi = k \left[2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} - 2x_1 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

Cusp

$$\pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 3(x_1^2 - t_{2n-5}) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

Swallowtail

$$\pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + 3(4x_1^3 + 2t_{2n-5}x_1 + t_{2n-4}) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

Butterfly

$$\pi = k \left[-2x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + (5x_1^4 + 3t_{2n-5}x_1^2 + 2t_{2n-4}x_1 + t_{2n-3}) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right]$$

The following equations depend on a real parameter s . Near a singularity locally modeled by the b_s, m_s, f_s , and w_s the corresponding Poisson bivectors are

Map b_s

$$\pi_s = k \left[2x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - 3(s - t_{2n-3}^2 + x_1^2) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right]$$

Map m_s

$$\pi_s = k \left[2x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - 3(s - t_{2n-3}^2 - x_1^2) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right]$$

Map f_s

$$\pi_s = k \left[2x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - (t_{2n-3} - 2sx_1 + 4x_1^3) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right]$$

Map w_s

$$\begin{aligned} \pi_s = k \left[(-2sx_2 - 4t_{2n-3}x_2 - 4x_1x_3) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + (-4x_1x_2 + 2sx_3 + 4t_{2n-3}x_3) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} \right. \\ \left. + (4x_2^2 + 4x_3^2) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial t_{2n-3}} - (2st_{2n-3} + 4t_{2n-3}^2 + 4x_1^2) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right. \\ \left. + 4(x_1x_2 - t_{2n-3}x_3) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial t_{2n-3}} - 4(t_{2n-3}x_2 + x_1x_3) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial t_{2n-3}} \right] \end{aligned}$$

The proof of this last result is included in appendix A.

4. Symplectic forms on leaves of generalized wrinkled fibrations

4.1. General criterion for constructing symplectic forms on leaves near the singularities

Theorem 4.1. *Under the hypothesis of Proposition 3.1, the symplectic form induced by the Poisson structure π on the symplectic leaf Σ_q through $q \neq 0$ is completely determined by the Poisson structure of the map f_o . That is, if u_q, v_q are tangent vectors to the leaves, then:*

$$\omega_{\Sigma_q}(u_q, v_q) = \omega_o(\tilde{u}_q, \tilde{v}_q)$$

where ω_o is the symplectic structure of f_o , and \tilde{u}_q, \tilde{v}_q are the tangent vectors u_q and v_q restricted to the last 4 coordinates.

Proof. First, we have to obtain vectors tangent to the leaves. That is, we want to find vectors such that they are annihilated simultaneously by the $2n - 2$ Casimir functions. Then we transpose the matrix and compute its null space.

In the case when f is a complex map, we used its real and imaginary parts of each coordinate function as Casimir functions. We obtained the matrix (3.1) whose transpose matrix is:

$$D^T = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & \frac{\partial C_{2n-3}}{\partial t_{2n-3}} & \frac{\partial C_{2n-3}}{\partial x_1} & \frac{\partial C_{2n-3}}{\partial x_2} & \frac{\partial C_{2n-3}}{\partial x_3} \\ 0 & \cdots & 0 & \frac{\partial C_{2n-2}}{\partial t_{2n-3}} & \frac{\partial C_{2n-2}}{\partial x_1} & \frac{\partial C_{2n-2}}{\partial x_2} & \frac{\partial C_{2n-2}}{\partial x_3} \end{pmatrix}$$

Note that its left upper block is an identity matrix of dimension $2n - 4$.

Let

$$\begin{aligned} \partial C_{2n-3} : &= \left(0, \dots, 0, \frac{\partial C_{2n-3}}{\partial t_{2n-3}}, \frac{\partial C_{2n-3}}{\partial x_1}, \frac{\partial C_{2n-3}}{\partial x_2}, \frac{\partial C_{2n-3}}{\partial x_3} \right) \\ \partial C_{2n-2} : &= \left(0, \dots, 0, \frac{\partial C_{2n-2}}{\partial t_{2n-3}}, \frac{\partial C_{2n-2}}{\partial x_1}, \frac{\partial C_{2n-2}}{\partial x_2}, \frac{\partial C_{2n-2}}{\partial x_3} \right) \end{aligned}$$

Then a vector $a = (a_1, a_2, \dots, a_{2n})$ belongs to $Ker(D^T)$ if and only if:

$$\begin{aligned} \langle \partial C_{2n-3}, a \rangle &= 0 \\ \langle \partial C_{2n-2}, a \rangle &= 0 \end{aligned}$$

Observe that the first $2n - 4$ entries of a equal zero. Then, $a \in Ker(D^T)$ if

$$a = (0, 0, \dots, 0, a_{2n-3}, a_{2n-2}, a_{2n-1}, a_{2n}),$$

where the coefficients $a_{2n-3}, a_{2n-2}, a_{2n-1}, a_{2n}$ are determined by the equations:

$$(4.1) \quad \begin{cases} a_{2n-3} \frac{\partial C_{2n-3}}{\partial t_{2n-3}} + a_{2n-2} \frac{\partial C_{2n-3}}{\partial x_1} + a_{2n-1} \frac{\partial C_{2n-3}}{\partial x_2} + a_{2n} \frac{\partial C_{2n-3}}{\partial x_3} = 0 \\ a_{2n-3} \frac{\partial C_{2n-2}}{\partial t_{2n-3}} + a_{2n-2} \frac{\partial C_{2n-2}}{\partial x_1} + a_{2n-1} \frac{\partial C_{2n-2}}{\partial x_2} + a_{2n} \frac{\partial C_{2n-2}}{\partial x_3} = 0 \end{cases}$$

Since the rank of the matrix D is $2n - 2$, it has nullity 2. Therefore there exist two vectors u_q and v_q that generate all solutions to the previous system. We may assume they are orthogonal. Now, we have to find vectors α_q, β_q such that $\mathcal{B}_q(\alpha_q) = u_q$ and $\mathcal{B}_q(\beta_q) = v_q$.

To compute the symplectic form it is enough to find α_q . In order to compute β_q we may proceed similarly. We know that α is the solution to the equation $\mathcal{B}(\alpha)(\cdot) = \pi(\cdot, \alpha) = u_q$.

It is equivalent to consider the system $\pi \cdot \alpha_q = u_q$ and solve for α_q . By the previous discussion and recalling the form of the Poisson matrix, if u_q, α_q and v_q have coordinates:

$$\begin{aligned} u_q &= (0, 0, \dots, u_{2n-3}, u_{2n-2}, u_{2n-1}, u_{2n}) \\ v_q &= (0, 0, \dots, u_{2n-3}, v_{2n-2}, v_{2n-1}, v_{2n}) \\ \alpha_q &= (\alpha_1, \alpha_2, \dots, \alpha_{2n}) \end{aligned}$$

This system is reduced to:

$$(4.2) \quad \begin{cases} u_{2n-3} &= \alpha_{2n-2} \pi^{12} + \alpha_{2n-1} \pi^{13} + \alpha_{2n} \pi^{14} \\ u_{2n-2} &= -\alpha_{2n-3} \pi^{12} + \alpha_{2n-1} \pi^{23} + \alpha_{2n} \pi^{24} \\ u_{2n-1} &= -\alpha_{2n-3} \pi^{13} - \alpha_{2n-2} \pi^{23} + \alpha_{2n} \pi^{34} \\ u_{2n} &= -\alpha_{2n-3} \pi^{14} - \alpha_{2n-2} \pi^{24} - \alpha_{2n-1} \pi^{34} \end{cases}$$

Therefore the symplectic form will be given by

$$\omega_{\Sigma_q}(q) = \langle \alpha_q, v_q \rangle,$$

here α_q is the solution to the system (4.2), and v satisfies the system (4.1). Note that we may choose α with the first $2n - 4$ coordinates equal zero.

When the map f is real we obtained the matrix (3.2). Its transpose is:

$$D^T = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial C_{2n-2}}{\partial t_1} & \dots & \frac{\partial C_{2n-2}}{t_{2n-4}} & \frac{\partial C_{2n-2}}{t_{2n-3}} & \frac{\partial C_{2n-2}}{x_1} & \frac{\partial C_{2n-2}}{x_2} & \frac{\partial C_{2n-2}}{x_3} \end{pmatrix}$$

Its left upper block is an identity matrix of dimension $2n - 3$. Then $a \in \text{Ker}(D^T)$ if $a = (0, 0, \dots, 0, 0, a_{2n-2}, a_{2n-1}, a_{2n})$, where the coefficients $a_{2n-2}, a_{2n-1}, a_{2n}$ are determined by the equation:

$$a_{2n-2} \frac{\partial C_{2n-2}}{x_1} + a_{2n-1} \frac{\partial C_{2n-2}}{x_2} + a_{2n} \frac{\partial C_{2n-2}}{x_3} = 0$$

We can give the explicit solutions, they are generated by the vectors:

$$(4.3) \quad u = \{0, 0, \dots, 0, -\frac{\frac{\partial C_{2n-2}}{x_2}}{\frac{\partial C_{2n-2}}{x_1}}, 1, 0\}, \quad v = \{0, 0, \dots, 0, -\frac{\frac{\partial C_{2n-2}}{x_3}}{\frac{\partial C_{2n-2}}{x_1}}, 0, 1\}$$

Let $u_q = u$ and $v_q = \text{proj}_u(v)$, the orthogonal projection of v over u . Then u_q and v_q are orthogonal and generate all solutions to the previous system. As before, we know that α_q is the solution to the equation $\mathcal{B}(\alpha)(\cdot) = \pi(\cdot, \alpha) = u_q$.

This is equivalent to solving the system $\pi \cdot \alpha_q = u_q$ for α_q . If α_q has coordinates:

$$\alpha_q = (\alpha_1, \alpha_2, \dots, \alpha_{2n})$$

this system is reduced to

$$(4.4) \quad \begin{cases} -\frac{\frac{\partial C_{2n-2}}{x_2}}{\frac{\partial C_{2n-2}}{x_1}} &= -\alpha_{2n-3}\pi^{12} + \alpha_{2n-1}\pi^{23} + \alpha_{2n}\pi^{24} \\ 1 &= -\alpha_{2n-3}\pi^{13} - \alpha_{2n-2}\pi^{23} + \alpha_{2n}\pi^{34} \\ 0 &= -\alpha_{2n-3}\pi^{14} - \alpha_{2n-2}\pi^{24} - \alpha_{2n-1}\pi^{34} \end{cases}$$

Therefore the symplectic form will be given by

$$\omega_{\Sigma_q}(q) = \langle \alpha_q, v_q \rangle$$

where α_q is the solution to the system (4.4), and v_q has the form (4.3). Note that we may choose α with the first $2n - 4$ coordinates equal zero. \square

4.2. Symplectic forms on the leaves of generalized wrinkled fibrations in dimension 6

As a corollary of the previous theorem we obtain the following result in dimension 6.

Corollary 4.2. . *Let M be a closed, orientable, smooth 6-manifold equipped with a generalized wrinkled fibration $f: M \rightarrow X$ on a smooth 4-manifold X . Let $(U, (t_1, t_2, t_3, x_1, x_2, x_3))$ be a coordinate neighbourhood of $q \in \text{Crit}_f$, an element of the singularity set of f . Then, there is a symplectic form on U induced by π on the symplectic leaf Σ_q through q near each of the singularities of the fibration with the following expressions:*

Indefinite Fold

$$(4.5) \quad \omega_{\Sigma_q} = \frac{x_1^2}{2k(q)(x_1^2 + x_3^2)^{1/2}} \omega_{Area}(q)$$

where ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^6 .

Definite Folds

For the definite fold singularities described by the equations (3.4) and (3.5) we obtain the symplectic forms

$$(4.6) \quad \omega_{\Sigma_q} = -\frac{x_1^2}{2(x_1^2 + x_3^2)^{1/2}} \omega_{Area}(q)$$

and

$$(4.7) \quad \omega_{\Sigma_q} = \frac{x_1^2}{2(x_1^2 + x_3^2)^{1/2}} \omega_{Area}(q)$$

respectively.

Indefinite Cusp

$$(4.8) \quad \omega_{\Sigma_q} = \frac{3x_2(t_1 - x_1^2)}{k(q)(9(t_1 - x_1^2)^2 + 4x_3^2)^{1/2}} \omega_{Area}(q)$$

where ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^6 .

Definite Cusps

The definite singularities modelled by the parametrizations (3.9) and (3.10) have the corresponding symplectic form which coincides in both cases :

$$(4.9) \quad \omega_{\Sigma_q} = \frac{3(t_1 - x_1^2)x_2}{(9(t_1 - x_1^2)^2 + 4x_3^2)^{1/2}} \omega_{Area}(q)$$

Indefinite Swallowtail

$$(4.10) \quad \omega_{\Sigma_q} = -\frac{t_2 + 2t_1x_1 + 4x_1^3}{k(q)((t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q)$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^6 .

Definite Swallowtail

The definite swallowtails modelled by the parametrizations (3.14) and (3.14) have the corresponding symplectic form which coincides in both cases:

$$(4.11) \quad \omega_{\Sigma_q} = -\frac{(t_2 + 2t_1x_1 + 4x_1^3)}{((t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q)$$

Indefinite Butterfly

$$(4.12) \quad \omega_{\Sigma_q} = -\frac{t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3)}{k(q)((t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}}\omega_{Area}(q)$$

where ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^6 .

Definite Butterfly

$$(4.13) \quad \omega_{\Sigma_q} = -\frac{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))}{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}}\omega_{Area}(q)$$

A proof may be found in appendix A.

4.3. Symplectic forms on higher dimensional type $2n$ generalized wrinkled fibrations

Corollary 4.3. For a non-vanishing smooth function $k \in C^\infty(M)$ we have the following consequences:

(1) Let M be a closed smooth oriented and connected $2n$ -manifold, and $f : M \rightarrow X$ a generalized broken Lefschetz fibration. The symplectic forms induced by the corresponding Poisson structures on the symplectic leaves Σ_q through a point $q = (t_1, \dots, t_{2n-3}, x_1, x_2, x_3)$ have the following local expressions:

Lefschetz-type singularity

Let $q \in B^{2n} \setminus \{0\}$. Near Lefschetz-type singularities the symplectic form is given by

$$\omega_{\Sigma_q} = \frac{1}{k(q)(t_{2n-3}^2 + x_1^2 + x_2^2 + x_3^2)}\omega_{Area}(p)$$

Indefinite fold singularity

Near indefinite fold singularities Z the symplectic form is locally described by

$$\omega_{\Sigma_q} = \frac{1}{k(q)\sqrt{x_1^2 + x_2^2 + x_3^2}}\omega_{Area}(q)$$

where $\omega_{Area}(q)$ is the area form on Σ_q induced by the metric

$$ds^2 = dt_1^2 + \dots + dt_{2n-3}^2 + dx_1^2 + dx_2^2 + dx_3^2$$

on $Z \times B^3$.

(2) Let M be a closed, orientable, smooth $2n$ -manifold endowed with a type $2n$ -wrinkled fibration f to a closed $2n-2$ manifold X . Let $q \in B^{2n} \setminus \{0\}$. Then the symplectic forms associated to the complete Poisson structure are given by the following expressions near the corresponding singularities:

Fold

$$\omega_{\Sigma_q} = \frac{x_1^2}{2k(q)(x_1^2 + x_3^2)^{1/2}} \omega_{Area}(q)$$

Cusp

$$\omega_{\Sigma_q} = \frac{3x_2(t_{2n-5} - x_1^2)}{k(q)(9(t_1 - x_1^2)^2 + 4x_3^2)^{1/2}} \omega_{Area}(q)$$

Swallowtail

$$\omega_{\Sigma_q} = -\frac{t_{2n-4} + 2t_{2n-5}x_1 + 4x_1^3}{k(q)((t_{2n-4} + 2t_{2n-5}x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}} \omega_{Area}(q)$$

Butterfly

$$\omega_{\Sigma_q} = -\frac{t_{2n-3} + x_1(2t_{2n-4} + 3t_{2n-5}x_1 + 5x_1^3)}{k(q)((t_{2n-3} + x_1(2t_{2n-4} + 3t_{2n-5}x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}} \omega_{Area}(q)$$

The following equations depend on a real parameter s . Near a singularity locally modeled by the maps b_s (3.23), m_s (3.24), f_s (3.25), and w_s (3.26) the corresponding symplectic forms are

Map b_s

$$\omega_{\Sigma_q} = \frac{(s - t_{2n-3}^2 + x_1^2)}{k(q)((s - t_{2n-3}^2 + x_1^2)^2(9(s - t_{2n-3}^2 + x_1^2)^2 + 4(x_2^2 + x_3^2)))^{1/2}} \omega_{Area}(q)$$

Map m_s

$$\omega_{\Sigma_q} = -\frac{(s - t_{2n-3}^2 - x_1^2)}{k(q)((s - t_{2n-3}^2 - x_1^2)^2(9(s - t_{2n-3}^2 - x_1^2)^2 + 4(x_2^2 + x_3^2)))^{1/2}} \omega_{Area}(q)$$

Map f_s

$$\omega_{\Sigma_q} = \frac{(t_{2n-3} - 2sx_1 + 4x_1^3)}{k(q)((t_{2n-3} - 2sx_1 + 4x_1^3)^2((t_{2n-3} - 2sx_1 + 4x_1^3)^2 + 4(x_2^2 + x_3^2)))^{1/2}} \omega_{Area}(q)$$

Map w_s

$$\omega_{\Sigma_q} = \frac{1}{2\mu k(q)} \cdot \frac{(t_{2n-3}x_2 + x_1x_3)((st_{2n-3} + 2(t_{2n-3}^2 + x_1^2))^2 + (x_3(s + 2t_{2n-3}) - 2x_1x_2)^2 + 4(t_{2n-3}x_2 + x_1x_3))}{((st_{2n-3} + 2(t_{2n-3}^2 + x_1^2))^2 + (x_3(s + 2t_{2n-3}) - 2x_1x_2)^2 + 4(t_{2n-3}x_2 + x_1x_3)^2)^{1/2}} \omega_{Area}(q)$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^{2n} , and

$$\begin{aligned} \mu^2 = & (t_{2n-3}x_2 + x_1x_3)^2(s^2(t_{2n-3}^2 + x_2^2 + x_3^2) + 4st_{2n-3}(t_{2n-3}^2 + x_1^2 + x_2^2 + x_3^2) + 4(t_{2n-3}^2 + x_1^2 + x_2^2 + x_3^2)^2) \\ & (s^2(t_{2n-3}^2 + x_3^2) + 4(t_{2n-3}^2 + x_1^2)(t_{2n-3}^2 + x_1^2 + x_2^2 + x_3^2) + 4s(t_{2n-3}^3 - x_1x_2x_3 + t_{2n-3}(x_1^2 + x_3^2))). \end{aligned}$$

For all these cases $\omega_{Area}(q)$ is the area form induced by the euclidean metric on B^{2n} .

We include a proof in appendix A.

5. Near-symplectic Forms on Generalized Wrinkled Fibrations

5.1. Near-symplectic Manifolds

We follow the definition of near-symplectic forms in higher dimensions as in [19]. Let M be an oriented manifold of dimension $2n$, and consider a 2-form $\omega \in \Omega^2(M)$ such that it is near-positive everywhere, that is $\omega^n \geq 0$. Denote by $K_p = \{v \in T_p M \mid \omega_p(v, \cdot) = 0\}$ the kernel of ω at a point $p \in M$. The collection of fibrewise kernels form the kernel of the 2-form $K := \ker(\omega) \subset TM$. If ω is symplectic, then $K_p = 0$. We relax the non-degeneracy condition by near-positive forms, we can consider non-trivial kernels K_p . There is an intrinsic gradient $\nabla_p \omega: K_p \rightarrow \Lambda^2 T_p^* M$. Restricting the gradient to bivectors in K_p results in a linear map

$$D_K := \nabla_p \omega|_K: K_p \rightarrow \Lambda^2 K^*.$$

The image $\text{Im}(D_K)$ has dimension at most 3. Assuming that K is 4-dimensional and $\text{Rank}(D_K) = 3$ it has been shown in [19] that the zero set of ω^{n-1} is a submanifold of M of dimension $2n - 3$.

Definition 5.1. A 2-form $\omega \in \Omega^2(M^{2n})$ is near-symplectic, if it is closed, $\omega^n \geq 0$, and at a point p where $\omega^n = 0$, one has that the kernel K is 4-dimensional and that the image $\text{Im}(D_K)$ has dimension 3.

The set $Z_\omega = \{p \in M \mid \omega_p^{n-1} = 0\}$ is called the singular locus of ω and it is a submanifold of codimension 3.

Remark 5.2. In dimension 6, the definition of a near-symplectic form implies that $\omega \in \Omega^2(M)$ is closed and for every $p \in M$, either

- (i) $\omega_p^3 > 0$ on $M \setminus Z_\omega$, or
- (ii) $\omega_p^2 = 0$ on a 3-submanifold Z_ω .

◇

Locally, a Darboux-type theorem for near-symplectic forms tells us that we can find a coordinate neighbourhood U around a point $p \in Z_\omega \subset (M, \omega)$ such that ω looks like the sum of a symplectic form of rank $2n - 4$ and a 4-dimensional near-symplectic form. On $(U, (z, x))$ with coordinates $z = (z_0, \dots, z_{2n-3})$ on Z_ω and normal coordinates $x = (x_1, x_2, x_3)$, we can express ω locally as

$$\begin{aligned} \omega &= \omega_Z - 2x_1(dz_0 \wedge dx_1 + dx_2 \wedge dx_3) + x_2(dz_0 \wedge dx_2 - dx_1 \wedge dx_3) + x_3(dz_0 \wedge dx_3 + dx_1 \wedge dx_2) \\ &= \omega_Z - 2x_1(\beta_1) + x_2(\beta_2) + x_3(\beta_3) \end{aligned}$$

where $\omega_Z := i^* \omega$ is a closed 2-form of maximal rank on Z_ω . On a 6-manifold, ω_Z would be of rank 2. The 2-forms β_1, β_2 and β_3 correspond to elements of a basis of the rank-3 bundle $\Lambda_+^2 \mathbb{R}^4$.

5.2. Proof of Theorem 1.3

Theorem 5.3. Let M be a closed oriented 6-manifold, (X, ω_X) a closed symplectic 4-manifold, and $f: M \rightarrow X$ a generalized wrinkled fibration. Denote by Z the singularity set of f , a 3-submanifold of M . Assume that there is a class $\alpha \in H^2(M)$, such that it pairs positively with every component of every fibre, and $\alpha|_Z = [\omega_X|_Z]$. Then there exist a near-symplectic form ω on M with singular locus Z such that it restricts to a symplectic form on the smooth fibres of the fibration.

Proof. The global construction of a near-symplectic form on a generalized wrinkled fibration is similar to the 4-dimensional case. Constructing a near-symplectic form on the total space of a broken Lefschetz fibration involves four steps [1] that extend to the case of a wrinkled fibration [14]. These steps appear again in the higher dimensional situation with generalized BLFs [19]. We briefly recall them. Step 1 constructs a local near-symplectic that is positive on the fibres. Steps 2 and 3 extend the 2-form to the neighbourhood of the fibres and then to the whole manifold using the cohomological assumptions of the theorem. Finally, step 4 involves Thurston's argument

to guarantee positivity on vertical and tangent subspaces. All these steps apply in the same way for generalized wrinkled fibrations. The only modification involves the local model of the 2-form around the new singularities. Once this is done, there is no difference anymore in the global construction. Since this adjustment applies to step 1, we give the local near-symplectic forms for each singularity.

5.3. Constructing near-symplectic forms. General Scheme

We begin by giving the general scheme to construct local near-symplectic forms around the critical set of f without coordinates. The specific formulæ in coordinates will be provided afterwards. To start, consider the following 2-form

$$(5.1) \quad \omega_0 = f^*\omega_X + *[f^*\omega_X^2] = du \wedge ds + dt \wedge df_4 + *(du \wedge ds \wedge dt \wedge df_4)$$

where $*$: $\Omega^4 M \rightarrow \Omega^2 M$ denotes the Hodge operator with respect to a Riemannian metric g on M , and df_4 is the 1-form defined by the fourth component of the generalized wrinkled fibration which varies according to the parametrization of each singularity. This 2-form is positive on the fibres and non-degenerate outside the singularity set by construction. The positivity on the fibres follows from $*[f^*\omega_X^2]$, since this 2-form is positive on the vertical subspaces, complementary to horizontal subspaces where the pullback $f^*\omega_X$ is positive. The non-degeneracy can be checked by looking at

$$\omega_0^3 = (f^*\omega_X)^3 + 3(f^*\omega_X)^2 \wedge *[f^*(\omega_X)] + 3(f^*\omega_X) \wedge *[f^*(\omega_X)^2] + *[f^*(\omega_X)^3].$$

Since $(f^*\omega_X)^3 = 0$ and $(*[f^*\omega_X])^2 = 0$, this 6-form reduces to $\omega_0^3 = 3\beta \wedge *\beta$ with $\beta = f^*\omega_X^2$, which is positive outside the singularity set. To transition to the description of the 2-forms in coordinates, first we notice that all our near-symplectic forms can be expressed in the following way:

$$\omega_0 = \omega_1 + f\omega_2 + g\omega_3 + h\omega_4$$

where $\omega_i \in \Omega^2(M)$, for $i = 1, 2, 3, 4$, and $f, g, h \in C^\infty(M)$ are determined by each singularity. In all cases we have $\omega_1 = du \wedge ds$, and up to an odd permutation and a minus sign,

$$\omega_2 = dt \wedge dx + dy \wedge dz, \quad \omega_3 = dt \wedge dy + dz \wedge dx, \quad \omega_4 = dt \wedge dz + dx \wedge dy.$$

Note that $\omega_1^2 = 0$, $\omega_2 \wedge \omega_3 = 0$, $\omega_2 \wedge \omega_4 = 0$, and $\omega_3 \wedge \omega_4 = 0$. Thus, we have

$$\omega_0^2 = f^2\omega_2^2 + g^2\omega_3^2 + h^2\omega_4^2$$

and

$$\omega_0^3 = f^2\omega_1 \wedge \omega_2^2 + g^2\omega_1 \wedge \omega_3^2 + h^2\omega_1 \wedge \omega_4^2 + f^3\omega_2^3 + g^3\omega_3^3 + h^3\omega_4^3.$$

This implies

$$\begin{aligned} \omega_0^3 &= f^2\omega_1 \wedge \omega_2^2 + g^2\omega_1 \wedge \omega_3^2 + h^2\omega_1 \wedge \omega_4^2 \\ &= (f^2\omega_1 + g^2\omega_1 + h^2\omega_1) \wedge (2dt \wedge dx \wedge dy \wedge dz) \\ &= (f^2 + g^2 + h^2)du \wedge ds \wedge dt \wedge dx \wedge dy \wedge dz. \end{aligned}$$

This form is clearly positive outside the singularity set. On each singularity we have that $\omega_0^3 = \omega_0^2 = 0$, since $df_4 = 0$ at Crit_f . At each critical point $p \in M$ we find a 4-dimensional kernel $K_p = \{v \in T_p M \mid \omega_p(v, \cdot) = 0\}$ spanned by $\langle \partial_t, \partial_x, \partial_y, \partial_z \rangle$, and the rank of $D_K: K_p \rightarrow \Lambda^2 K^*$ is three. With these properties ω_0 is near-positive, i.e. $\omega_0^3 \geq 0$, and it satisfies the transversality condition. The only condition we lack now for this form to be near-symplectic is for it to be closed.

The 2-form (5.1) is closed only around the fold singularities but not for the other three. Thus, we need to add a suitable 2-form η so that $\omega_A = \omega_0 + \eta$ is closed. Fix g on K , such that $\omega|_K$ is

self-dual. We can then define a rescaling map $R_\varepsilon: \Omega^2 K^* \rightarrow \Omega^2 K^*$ and apply it to ω_0 . Finally, we add a small ε to preserve the non-degeneracy.

$$(5.2) \quad \omega = R_\varepsilon(\omega_0) + \varepsilon \cdot \eta.$$

With a suitable choice of η , it can be checked that the near-positive properties of ω_0 are preserved and it is closed. Thus, the 2-form (5.2) provides the desired near-symplectic form.

5.4. Folds

Since this singularity is also present in generalized bLfs, the proof for this case follows exactly as in the proof of Theorem 1 in [19]. We will only recall a couple of useful facts that will be applied to the other singularities. The 4-dimensional kernel of the near-symplectic form is $K = \nu \oplus NZ$, where ν is the line bundle defined by $\nu = \ker(f^*\omega_X)$ and NZ is the normal bundle of the singular locus Z . Using the same coordinates parametrizing the folds we can express the 4-dimensional tangent subspace as $K = \text{span}\langle \partial_t, \partial_x, \partial_y, \partial_z \rangle$. Let $f_4(u, s, t, x, y, z) = \frac{1}{2}(x^2 + y^2) - z^2$. The local model described in step 1 of [19] can be expressed as:

$$(5.3) \quad \begin{aligned} \omega &= f^*\omega_X + *[(f^*(\omega_X^2))] \\ &= du \wedge ds + xdt \wedge dx + ydt \wedge dy - 2zdt \wedge dz \\ &\quad + *(xdu \wedge ds \wedge dt \wedge dx + ydu \wedge ds \wedge dt \wedge dy - 2zdu \wedge ds \wedge dt \wedge dz) \\ &= du \wedge ds + x(dt \wedge dx + dy \wedge dz) + y(dt \wedge dy + dz \wedge dx) - 2z(dt \wedge dz + dx \wedge dy) \end{aligned}$$

This 2-form is already near-symplectic so it does not require any rescaling nor additional terms and $\eta = 0$.

5.5. Cusps

A generalized wrinkled fibration has real and oriented coordinate charts around cusps with parametrization given by

$$f: (u, s, t, x, y, z) \mapsto (u, s, t, x^3 - 3t \cdot x + y^2 - z^2).$$

Following Lekili's scheme [14], we start with the 2-form

$$(5.4) \quad \begin{aligned} \omega_0 &= f^*\omega_X + *[(f^*(\omega_X^2))] \\ &= du \wedge ds + 3(x^2 - t)(dt \wedge dx + dy \wedge dz) + 2y(dt \wedge dy - dx \wedge dz) \\ &\quad - 2z(dt \wedge dz + dx \wedge dy). \end{aligned}$$

This 2-form ω_0 is near-positive, the kernel $K_p = \ker(\omega(p)) \subset T_p M$ is 4-dimensional spanned by $\langle \partial_t, \partial_x, \partial_y, \partial_z \rangle$, and the rank of D_K at the singular points is 3. This form is not closed though, as $d\omega_0 = 6xdx \wedge dy \wedge dz - 3dt \wedge dy \wedge dz$. We modify it, and add the 2-form $\eta = -6xy(dz \wedge dy) - 3y(dt \wedge dx)$. To preserve the positivity on the fibres we introduce a scaling map. Locally we have splitting $T_p M = K_p \oplus \text{Symp}_{Z_p}$. Equipping M with a Riemannian metric g , we can restrict $g_K := g|_K$ such that $\omega|_K$ is self-dual, and consider the Hodge-* operator $*_{g_K}: \Omega^2 K^* \rightarrow \Omega^2 K^*$. Thus we can define a scaling map $R_\varepsilon: \Omega_+^2 K^* \rightarrow \Omega_+^2 K^*$ on basis elements of the space of self-dual forms on K :

$$\begin{aligned} R_\varepsilon(dt \wedge dx + dy \wedge dz) &= \varepsilon(dt \wedge dx + dy \wedge dz) \\ R_\varepsilon(dt \wedge dy + dz \wedge dx) &= dt \wedge dy + dz \wedge dx \\ R_\varepsilon(dt \wedge dz + dx \wedge dy) &= dt \wedge dz + dx \wedge dy \end{aligned}$$

Applying R_ε , we now find the near symplectic form ω adapted to a neighborhood of the cusp singularity, as intended

$$\omega = R_\varepsilon(\omega_0) + \varepsilon \cdot \eta = du \wedge ds + R_\varepsilon(dt \wedge df_4 + *_{g_K}(dt \wedge df_4)) + \varepsilon \cdot \eta.$$

Expanding the previous expression in coordinates we obtain

$$(5.5) \quad \begin{aligned} \omega = & du \wedge ds + 3\varepsilon(x^2 - t)(dt \wedge dx + dy \wedge dz) + 2ydt \wedge dy + (2y - 6\varepsilon xy)dz \wedge dx \\ & - (2z + 3\varepsilon y)dt \wedge dz - 2zdx \wedge dy. \end{aligned}$$

A basis for the tangent space of the fibre is given by the vectors:

$$\begin{aligned} v_1 &= \left(\frac{2z}{3(x^2 - t)} \right) \frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \\ v_2 &= \left(\frac{2y}{3(x^2 - t)} \right) \frac{\partial}{\partial x} - \frac{\partial}{\partial y}. \end{aligned}$$

By evaluating the 2-form ω on tangent vectors to the fibres we can see that for a sufficiently small ε the form (5.5) is positive on the fibres of f (see appendix B).

5.6. Swallowtails

The coordinate charts around a swallowtail are given by

$$f: (u, s, t, x, y, z) \mapsto (u, s, t, x^4 + sx^2 + tx + y^2 - z^2).$$

Define our initial form ω_0 to be

$$\begin{aligned} \omega_0 &= f^*\omega_X + *[(f^*(\omega_X^2))] \\ &= du \wedge ds + (4x^3 + 2sx + t)(dt \wedge dx + dx \wedge dy) + 2y(dt \wedge dy - dx \wedge dz) \\ &\quad - 2z(dt \wedge dz + dx \wedge dy). \end{aligned}$$

This form is non-degenerate outside the critical set and evaluates positively on the fibres of f . At a critical point $p \in M$ we have a splitting $T_p M = K_p \oplus \text{Symp}_Z$, where $K = \text{span}\langle \partial_t, \partial_x, \partial_y, \partial_z \rangle$ and $\text{Symp}_Z \subset TZ$ is the symplectic subspace given by $du \wedge ds$.

However, this 2-form is not closed. Thus, we add the following extra terms to ω_0 ,

$$\begin{aligned} \eta = & -2zdt \wedge dy + (12x^2 - 2s)ydz \wedge dx - ydt \wedge dz - (12x^2 - 2s)2zdx \wedge dy \\ & - x^2dt \wedge ds - 2yzds \wedge dx + 2xzds \wedge dy \end{aligned}$$

and obtain $\omega = \omega_0 + \eta$. This 2-form is now closed. To preserve the non-degeneracy, we multiply ω_0 by the function R_ε and the 2-form η by ε .

(5.6)

$$\begin{aligned} \omega = & R_\varepsilon(\omega_0) + \varepsilon \cdot \eta \\ = & du \wedge ds + \varepsilon(4x^3 + 2sx + t)(dt \wedge dx + dx \wedge dy) + 2y(dt \wedge dy - dx \wedge dz) - 2z(dt \wedge dz + dx \wedge dy) \\ & + \varepsilon[-2zdt \wedge dy + (12x^2 - 2s)ydz \wedge dx - ydt \wedge dz - (12x^2 - 2s)2zdx \wedge dy \\ & - x^2dt \wedge ds - 2yzds \wedge dx + 2xzds \wedge dy] \end{aligned}$$

This 2-form is closed, non-degenerate outside the singularity, and at the singular points it has a 4-dimensional kernel and $\text{Rank}(D_K) = 3$. Thus, this is a near-symplectic form defined on a small neighbourhood around a swallowtail point.

Using the basis of vectors tangent to the fibre given by

$$\begin{aligned} v_1 &= \left(\frac{2z}{4x^3 + 2sx + t} \right) \frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \\ v_2 &= \left(\frac{2y}{4x^3 + 2sx + t} \right) \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \end{aligned}$$

we can see that for a sufficiently small ε the previous 2-form ω is positive on the fibres of f (see appendix B).

5.7. Butterflies

The local model of generalized wrinkled fibration around a butterfly point is

$$f: (u, s, t, x, y, z) \mapsto (u, s, t, x^5 + ux^3 + sx^2 + tx + y^2 - z^2).$$

Following the same scheme as for the other singularities, we begin with the 2-form

$$\begin{aligned} \omega_0 &= f^* \omega_X + * [(f^*(\omega_X^2))] \\ &= du \wedge ds + (5x^4 - 3ux^2 + 2sx - t)(dt \wedge dx + dx \wedge dy) + 2y(dt \wedge dy - dx \wedge dz) \\ &\quad - 2z(dt \wedge dz + dx \wedge dy) - x^3 dt \wedge du + x^2 dt \wedge ds. \end{aligned}$$

This form has a 4-dimensional kernel K at the singular points and the rank of D_K is 3. It is also non-degenerate outside the critical set and evaluates positively on the fibres of f , but it is not closed. By adding

$$\begin{aligned} \eta &= (-10x^3 + 3ux - s)(4zdx \wedge dy + 2ydx \wedge dz) + 3x^2(2du \wedge dz - ydu \wedge dz) \\ &\quad + ydt \wedge dz + 2zdt \wedge dy - 2xyds \wedge dz - 4xzds \wedge dy \end{aligned}$$

then the 2-form $\omega_0 + \eta$ is closed. To preserve the non-degeneracy, we add a scaling factor and obtain the local near-symplectic form around the butterfly point $\omega = R_\varepsilon(\omega_0) + \varepsilon \cdot \eta$. In coordinates this is

$$\begin{aligned} \omega &= du \wedge ds + \varepsilon \cdot (5x^4 - 3ux^2 + 2sx - t)(dt \wedge dx + dx \wedge dy) + 2y(dt \wedge dy - dx \wedge dz) \\ &\quad - 2z(dt \wedge dz + dx \wedge dy) - \varepsilon \cdot x^3 dt \wedge du + \varepsilon \cdot x^2 dt \wedge ds \\ &\quad + \varepsilon[(-10x^3 + 3ux - s)(4zdx \wedge dy + 2ydx \wedge dz) + 3x^2(2du \wedge dz - ydu \wedge dz) \\ &\quad + ydt \wedge dz + 2zdt \wedge dy - 2xyds \wedge dz - 4xzds \wedge dy]. \end{aligned}$$

A basis of tangent vectors to the fibres is given by

$$\begin{aligned} v_1 &= \left(0, 0, 0, \frac{2z}{5x^4 - 3ux^2 + 2sx - t}, 0, 1\right) \quad \text{and} \\ v_2 &= \left(0, 0, 0, \frac{2y}{5x^4 - 3ux^2 + 2sx - t}, -1, 0\right). \end{aligned}$$

For a sufficiently small ε , the 2-form ω is positive on the fibres of f (see appendix B). □

Appendix A. Computations of local expressions

Here we present the details concerning the calculations obtained in Corollaries 3.3, 4.2, and 4.3. This might allow for an easier verification of the results.

A.1. Local expressions for the Poisson structures

Local expressions near a fold singularity

The local coordinate model around a fold singularity is given by the map:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, -x_1^2 + x_2^2 + x_3^2)$$

Considering each coordinate function as a Casimir function for the Poisson bivector that we want to find, we compute the differential matrix of the map

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2x_1 \\ 0 & 0 & 0 & 2x_2 \\ 0 & 0 & 0 & 2x_3 \end{pmatrix}.$$

As we described, this gives a bivector matrix, which in this case is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2kx_3 & -2kx_2 \\ 0 & 0 & 0 & -2kx_3 & 0 & -2kx_1 \\ 0 & 0 & 0 & 2kx_2 & 2kx_1 & 0 \end{pmatrix}$$

Therefore the Poisson structure in the local coordinates of a fold singularity is described by equation 3.3.

We also compute the Poisson bivector for definite singularities for each wrinkled fibration. In this case, they are locally modeled by (3.4) and (3.5):

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^2 + x_2^2 + x_3^2)$$

Following the same computations as above, the Poisson matrix is then:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x_3 & -2x_2 \\ 0 & 0 & 0 & -2x_3 & 0 & 2x_1 \\ 0 & 0 & 0 & 2x_2 & -2x_1 & 0 \end{pmatrix}$$

It follows that the Poisson bivector is given by 3.6.

For the case when the map is

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^2 + x_2^2 + x_3^2).$$

the Poisson matrix is then:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2x_3 & 2x_2 \\ 0 & 0 & 0 & 2x_3 & 0 & -2x_1 \\ 0 & 0 & 0 & -2x_2 & 2x_1 & 0 \end{pmatrix}$$

Hence the Poisson bivector is given by 3.7.

Local expressions near a cusp singularity.

The local coordinate model around a cusp singularity is given by:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1x_1 + x_2^2 - x_3^2)$$

The differential matrix of the map is

$$\begin{pmatrix} 1 & 0 & 0 & -3x_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3x_1^2 - 3t_1 \\ 0 & 0 & 0 & 2x_2 \\ 0 & 0 & 0 & -2x_3 \end{pmatrix}$$

The corresponding bivector matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2kx_3 & -2kx_2 \\ 0 & 0 & 0 & 2kx_3 & 0 & 3k(x_1^2 - t_1) \\ 0 & 0 & 0 & 2kx_2 & 3k(t_1 - x_1^2) & 0 \end{pmatrix}.$$

Thus, the Poisson bivector in the local coordinates of a cusp singularity is given by [3.8](#).

For definite singularities in cusps, we obtain in each case [\(3.9\)](#) and [\(3.10\)](#):

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1x_1 + x_2^2 + x_3^2)$$

The Poisson matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x_3 & -2x_2 \\ 0 & 0 & 0 & -2x_3 & 0 & 3x_1^2 - 3t_1 \\ 0 & 0 & 0 & 2x_2 & 3t_1 - 3x_1^2 & 0 \end{pmatrix}.$$

Then the corresponding bivector is [3.11](#).

For the case:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^3 - 3t_1x_1 - x_2^2 - x_3^2)$$

The Poisson matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2x_3 & 2x_2 \\ 0 & 0 & 0 & -2x_3 & 0 & 3x_1^2 - 3t_1 \\ 0 & 0 & 0 & -2x_2 & 3t_1 - 3x_1^2 & 0 \end{pmatrix}$$

Therefore the Poisson bivector is given by the equation [3.12](#).

Local expressions near a swallowtail singularity

The local coordinate model around a swallowtail singularity is given by the map:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1x_1^2 + t_2x_1 + x_2^2 - x_3^2)$$

Its differential matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & x_1^2 \\ 0 & 1 & 0 & x_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4x_1^3 + 2t_1x_1 + t_2 \\ 0 & 0 & 0 & 2x_2 \\ 0 & 0 & 0 & -2x_3 \end{pmatrix}$$

The corresponding matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2kx_3 & -2kx_2 \\ 0 & 0 & 0 & 2kx_3 & 0 & k(4x_1^3 + 2t_1x_1 + t_2) \\ 0 & 0 & 0 & 2kx_2 & k(-4x_1^3 - 2t_1x_1 - t_2) & 0 \end{pmatrix}$$

It produces the Poisson bivector in the local coordinates of a swallowtail singularity described by equation 3.13.

For the corresponding definite singularities:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1x_1^2 + t_2x_1 + x_2^2 + x_3^2)$$

The Poisson matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x_3 & -2x_2 \\ 0 & 0 & 0 & -2x_3 & 0 & 4x_1^3 + 2t_1x_1 + t_2 \\ 0 & 0 & 0 & 2x_2 & -4x_1^3 - 2t_1x_1 - t_2 & 0 \end{pmatrix}$$

The Poisson bivector is 3.16.

In the case when the local form of the definite singularity is:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^4 + t_1x_1^2 + t_2x_1 - x_2^2 - x_3^2)$$

The Poisson matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2x_3 & 2x_2 \\ 0 & 0 & 0 & 2x_3 & 0 & 4x_1^3 + 2t_1x_1 + t_2 \\ 0 & 0 & 0 & -2x_2 & -4x_1^3 - 2t_1x_1 - t_2 & 0 \end{pmatrix}$$

The corresponding bivector is 3.17.

Local expressions near a butterfly singularity

The local coordinate model around a butterfly singularity is given by:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1x_1^3 + t_2x_1^2 + t_3x_1 + x_2^2 - x_3^2)$$

The differential of the map is:

$$\begin{pmatrix} 1 & 0 & 0 & x_1^3 \\ 0 & 1 & 0 & x_2^2 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 5x_1^4 + 3t_1x_1^2 + 2t_2x_1 + t_3 \\ 0 & 0 & 0 & 2x_2 \\ 0 & 0 & 0 & -2x_3 \end{pmatrix}$$

The corresponding matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2kx_3 & -2kx_2 \\ 0 & 0 & 0 & 2kx_3 & 0 & k(5x_1^4 + 3t_1x_1^2 + 2t_2x_1 + t_3) \\ 0 & 0 & 0 & 2kx_2 & k(-5x_1^4 - 3t_1x_1^2 - 2t_2x_1 - t_3) & 0 \end{pmatrix}$$

Then the Poisson bivector in the local coordinates of a butterfly singularity is described by [3.18](#).

For definite singularities:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1x_1^3 + t_2x_1^2 + t_3x_1 + x_2^2 + x_3^2)$$

The Poisson matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x_3 & -2x_2 \\ 0 & 0 & 0 & -2x_3 & 0 & 5x_1^4 + 3t_1x_1^2 + 2t_2x_1 + t_3 \\ 0 & 0 & 0 & 2x_2 & -5x_1^4 - 3t_1x_1^2 - 2t_2x_1 - t_3 & 0 \end{pmatrix}$$

Then the corresponding bivector is [3.21](#).

When the local form of the definite singularity is:

$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, x_1^5 + t_1x_1^3 + t_2x_1^2 + t_3x_1 - x_2^2 - x_3^2)$$

The Poisson matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2x_3 & 2x_2 \\ 0 & 0 & 0 & 2x_3 & 0 & 5x_1^4 + 3t_1x_1^2 + 2t_2x_1 + t_3 \\ 0 & 0 & 0 & -2x_2 & -5x_1^4 - 3t_1x_1^2 - 2t_2x_1 - t_3 & 0 \end{pmatrix}$$

Then the bivector is [3.22](#).

A.2. Equations for the symplectic forms on the leaves near singularities

Indefinite Fold

As we described in the general procedure, if u_q, v_q are tangent vectors to the leaves there exist co-vectors $\alpha_q, \beta_q \in T_q^*M$ such that $\mathcal{B}_q(\alpha_q) = u_q$ and $\mathcal{B}_q(\beta_q) = v_q$, where the map \mathcal{B}_q is given by:

$$\mathcal{B}_q(\alpha)(\cdot) = \pi_q(\cdot, \alpha)$$

Therefore, if we want to find two tangent vectors to the symplectic leaves we have to give vectors annihilated simultaneously by the differential of four Casimir functions for the corresponding Poisson structure.

A straightforward calculation yields that the vectors,

$$u_q = \frac{x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3}}{(x_1^2 + x_3^2)^{1/2}}$$

$$v_q = \frac{x_1^2 x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_1 x_2 x_3 \frac{\partial}{\partial x_3}}{(x_1^2 + x_3^2)^{1/2}}$$

are tangent to Σ_q at q , and orthogonal with respect to the euclidean metric

$$ds^2 = dt_1^2 + dt_2^2 + dt_3^2 + dx_1^2 + dx_2^2 + dx_3^2$$

on B^6 . Using the local expression of the Poisson structure for a fold singularity given by equation (3.3), one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{x_3 dx_1 + x_1 dx_2}{k(q)(x_1^2 + x_3^2)^{1/2}}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{-x_1 x_2 x_3 dx_2 - x_1(x_1^2 + x_3^2)}{2k(q)(x_1^2 + x_3^2)}.$$

A direct calculation now implies that the symplectic form is given by 4.5:

$$\omega_{\Sigma_q}(q)(u_q, v_q) = \langle \alpha_q, v_q \rangle = \frac{x_1^2}{2k(q)(x_1^2 + x_3^2)^{1/2}}$$

For definite singularities described by the equations (3.4) and (3.5) we obtain the symplectic forms

$$\omega_{\Sigma_q} = -\frac{x_1^2}{2(x_1^2 + x_3^2)^{1/2}} \omega_{Area}(q)$$

and

$$\omega_{\Sigma_q} = \frac{x_1^2}{2(x_1^2 + x_3^2)^{1/2}} \omega_{Area}(q)$$

respectively. This follows directly with the same computations of the previous case. Tangent vectors to the leaves u_q and v_q are slightly different, one component changes its sign. This creates a change of sign on one of the components of the corresponding vectors α_q and β_q .

Indefinite Cusps

In this case we find that the vectors,

$$u_q = \frac{-2x_3 \frac{\partial}{\partial x_1} + 3(t_1 - x_1^2) \frac{\partial}{\partial x_3}}{(9(t_1 - x_1^2)^2 + 4x_3^2)^{1/2}}$$

$$v_q = \frac{(2x_2 - 8x_2 x_3^2) \frac{\partial}{\partial x_1} + 3(t_1 - x_1^2)(9(t_1 - x_1^2)^2 + 4x_3^2) \frac{\partial}{\partial x_2} + 12(t_1 - x_1^2)x_2 x_3 \frac{\partial}{\partial x_3}}{9(t_1 - x_1^2)^2 + 4x_3^2}$$

are tangent to Σ_q at q , and orthogonal with respect to the euclidean metric

$$ds^2 = dt_1^2 + dt_2^2 + dt_3^2 + dx_1^2 + dx_2^2 + dx_3^2$$

on B^6 . Using the corresponding local expression of the bivector (3.8), we check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{3(t_1 - x_1^2)dx_1 + x_3dx_3}{2k(q)(9t_1^2 - 18t_1x_1^2 + 9x_1^4 + 4x_3^2)^{1/2}}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{6(t_1 - x_1^2)x_2x_3dx_1 - 9(t_1 - x_1^2)^2x_2dx_3}{k(q)(9(t_1 - x_1^2)^2 + 4x_3^2)}.$$

Now a direct calculation gives that the symplectic form is 4.8.

For the definite singularities modelled by the equations (3.9) and (3.10), the corresponding symplectic forms on the leaves coincide with the previous one:

$$\omega_{\Sigma_q} = \frac{3(t_1 - x_1^2)x_2}{(9(t_1 - x_1^2)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q)$$

This last equality follows from very similar computations as in the previous case, up to a sign, as in the fold case.

Indefinite Swallowtail

We find that the vectors,

$$\begin{aligned} u_q &= \frac{2x_3\frac{\partial}{\partial x_1} + (t_2 + 2t_1x_1 + 4x_1^3)\frac{\partial}{\partial x_3}}{((t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}} \\ v_q &= \frac{(-2x_2(t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2 + 8x_3^2)\frac{\partial}{\partial x_1} + (t_2 + 2t_1x_1 + 4x_1^3)\frac{\partial}{\partial x_2}}{(t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2} \\ &\quad + \frac{4(t_2 + 2t_1x_1 + 4x_1^3)x_2x_3\frac{\partial}{\partial x_3}}{(t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2} \end{aligned}$$

are tangent to Σ_q at q , and orthogonal with respect to the euclidean metric

$$ds^2 = dt_1^2 + dt_2^2 + dt_3^2 + dx_1^2 + dx_2^2 + dx_3^2$$

on B^6 . Using the local expression of the Poisson structure for a fold singularity given by equation (3.3), one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{(t_2 + 2t_1x_1 + 4x_1^3)dx_1 - 2x_3dx_3}{2x_2k(q)((t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\begin{aligned} \beta_q &= \frac{1}{k} \left(\frac{2x_3(t_2 + 2t_1x_1 + 4x_1^3)dx_1}{t_2^2 + 4t_1t_2x_1 + 4t_1^2x_1^2 + 8t_2x_1^3 + 16t_1x_1^4 + 16x_1^6 + 4x_3^2} \right. \\ &\quad \left. + \left(1 - \frac{4x_3^2}{(t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2} \right) dx_3 \right) \end{aligned}$$

A direct calculation now implies that the symplectic form is 4.10.

For the definite singularities we obtain that the symplectic forms on the leaves coincide in both cases (3.14) and (3.15):

$$\omega_{\Sigma_q} = -\frac{(t_2 + 2t_1x_1 + 4x_1^3)}{((t_2 + 2t_1x_1 + 4x_1^3)^2 + 4x_3^2)^{1/2}}\omega_{Area}(q)$$

Analogously to the last cases, we proceed changing the corresponding signs.

Butterfly singularity

We find that the vectors,

$$u_q = \frac{2x_3 \frac{\partial}{\partial x_1} + (t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3)) \frac{\partial}{\partial x_3}}{((t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}}$$

$$v_q = \left(\frac{8x_2x_3^2}{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2} - 2x_2 \right) \frac{\partial}{\partial x_1}$$

$$+ (t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3)) \frac{\partial}{\partial x_2}$$

$$+ \frac{4(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))x_2x_3}{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2} \frac{\partial}{\partial x_3}$$

are tangent to Σ_q at q , and orthogonal with respect to the euclidean metric

$$ds^2 = dt_1^2 + dt_2^2 + dt_3^2 + dx_1^2 + dx_2^2 + dx_3^2$$

on B^6 . Using the local expression of the Poisson structure for a fold singularity given by equation (3.3), one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{(t_3 + 2t_2x_1 + 3t_1x_1^2 + 5x_1^4)dx_1 - 2x_3dx_3}{2x_2k(q)((t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}}$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{2x_3(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))}{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2} dx_1$$

$$+ \left(1 - \frac{4x_3^2}{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2} \right) dx_3.$$

A direct calculation now implies that the symplectic form is given by 4.12:

We have that for the corresponding butterfly definite singularities the symplectic form is in both cases:

$$\omega_{\Sigma_q} = - \frac{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))}{(t_3 + x_1(2t_2 + 3t_1x_1 + 5x_1^3))^2 + 4x_3^2)^{1/2}} \omega_{Area}(q)$$

Appendix B. Positivity on fibres of local near-symplectic forms

Cusp:

Near-symplectic form:

$$\omega = du \wedge ds + 3\varepsilon(x^2 - t)(dt \wedge dx + dy \wedge dz) + 2ydt \wedge dy + (2y - 6\varepsilon xy)dz \wedge dx$$

$$- (2z + 3\varepsilon y)dt \wedge dz - 2zdx \wedge dy.$$

Map:

$$f: (u, s, t, x, y, z) \mapsto (u, s, t, x^3 - 3t \cdot x + y^2 - z^2)$$

Differential:

$$Df = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x & 3(x^2 - t) & 2y & -2z \end{pmatrix}$$

Basis of vectors tangent to the fibre:

$$v_1 = \left(0, 0, 0, \frac{2z}{3(x^2 - t)}, 0, 1\right)$$

$$v_2 = \left(0, 0, 0, \frac{2y}{3(x^2 - t)}, -1, 0\right)$$

Relevant terms of the 2-form when evaluated on v_1 and v_2 :

$$\tilde{\omega} = 3\varepsilon(x^2 - t)dy \wedge dz + (2y - 6\varepsilon xy)dz \wedge dx - 2zdx \wedge dy.$$

Evaluating $\omega(v_1, v_2)$, which amounts to evaluate $\tilde{\omega}(v_1, v_2)$ we obtain:

$$\begin{aligned} \omega(v_1, v_2) &= \tilde{\omega}(v_1, v_2) = \frac{1}{3(x^2 - t)} (3\varepsilon(x^2 - t)^2 + 4y^2 - 12\varepsilon xy^2 + 4z^2) \\ (B.1) \quad &= \frac{1}{3(x^2 - t)} (3\varepsilon(x^2 - t)^2 + 4y^2(1 - \varepsilon 3x) + 4z^2) \end{aligned}$$

All terms are always positive except possibly $12\varepsilon xy^2$. However, taking a sufficiently small neighbourhood with $|x| < 1$, this term is at most $12\varepsilon y^2$. By choosing ε to be sufficiently small we can arrange that this term will be smaller than $4y^2$, hence the whole expression remains positive.

Swallowtails:

Near-symplectic form:

$$\begin{aligned} \omega &= du \wedge ds + \varepsilon(4x^3 + 2sx + t)(dt \wedge dx + dy \wedge dz) + 2y(dt \wedge dy - dx \wedge dz) - 2z(dt \wedge dz + dx \wedge dy) \\ &\quad + \varepsilon[-2zdt \wedge dy + (12x^2 - 2s)ydz \wedge dx - ydt \wedge dz - (12x^2 - 2s)2zdx \wedge dy \\ &\quad - x^2dt \wedge ds - 2yzds \wedge dx + 2xzds \wedge dy] \end{aligned}$$

Map:

$$f: (u, s, t, x, y, z) \mapsto (u, s, t, x^4 + sx^2 - tx + y^2 - z^2)$$

Differential:

$$Df = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & x^2 & x & 4x^3 + 2sx - t & 2y & -2z \end{pmatrix}$$

Basis of vectors tangent to the fibre:

$$v_1 = \left(0, 0, 0, \frac{2z}{4x^3 + 2xs - t}, 0, 1\right)$$

$$v_2 = \left(0, 0, 0, \frac{2y}{4x^3 + 2xs - t}, -1, 0\right)$$

Relevant terms of the 2-form when evaluated on v_1 and v_2

$$\begin{aligned} \tilde{\omega} &= \varepsilon(4x^3 + 2sx - t)(dt \wedge dx + dy \wedge dz) + 2y(dt \wedge dy - dx \wedge dz) - 2z(dt \wedge dz + dx \wedge dy) \\ &\quad + \varepsilon[(12x^2 - 2s)ydz \wedge dx - (12x^2 - 2s)2zdx \wedge dy] \\ &= \varepsilon(4x^3 + 2sx - t)(dt \wedge dx + dy \wedge dz) + y[\varepsilon(12x^2 - 2s) + 2](dt \wedge dy - dx \wedge dz) \\ &\quad - 2z[\varepsilon(12x^2 - 2s) + 1](dt \wedge dz + dx \wedge dy) \end{aligned}$$

Evaluating $\omega(v_1, v_2)$, which amounts to evaluate $\tilde{\omega}(v_1, v_2)$, we obtain:

$$\begin{aligned}
\omega(v_1, v_2) &= \tilde{\omega}(v_1, v_2) = \\
\text{(B.2)} \quad &= \frac{1}{4x^3 + 2sx - t} (\varepsilon(4x^3 + 2sx - t)^2 + 2y^2[\varepsilon(12x^2 - 2s) + 2] + 4z^2(\varepsilon(12x^2 - 2s) + 1))
\end{aligned}$$

By restricting s to a sufficiently small interval around 0 which can be scaled by ε , the previous expression remains positive.

Butterfly:

Near-symplectic form:

$$\begin{aligned}
\omega &= du \wedge ds + \varepsilon(5x^4 - 3ux^2 + 2sx - t)(dt \wedge dx + dy \wedge dz) + 2y(dt \wedge dy - dx \wedge dz) \\
&\quad - 2z(dt \wedge dz + dx \wedge dy) - \varepsilon dt \wedge du + \varepsilon x^2 dt \wedge ds \\
&\quad + \varepsilon[(-10x^3 + 3ux - s)(4zdx \wedge dy + 2ydx \wedge dz) + 3x^2(2du \wedge dz - ydu \wedge dz) \\
&\quad + y dt \wedge dz + 2z dt \wedge dy - 2xy ds \wedge dz - 4xz ds \wedge dy]
\end{aligned}$$

Map:

$$f: (u, s, t, x, y, z) \mapsto (u, s, t, x^5 - ux^3 + sx^2 - tx + y^2 - z^2)$$

Differential:

$$Df = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -x^3 & x^2 & -x & 5x^4 - 3ux^2 + 2sx - t & 2y & -2z \end{pmatrix}$$

Basis of vectors tangent to the fibre:

$$\begin{aligned}
v_1 &= \left(0, 0, 0, \frac{2z}{5x^4 - 3ux^2 + 2sx - t}, 0, 1\right) \\
v_2 &= \left(0, 0, 0, \frac{2y}{5x^4 - 3ux^2 + 2sx - t}, -1, 0\right)
\end{aligned}$$

Relevant terms of the 2-form when evaluated on v_1 and v_2 :

$$\begin{aligned}
\tilde{\omega} &= \varepsilon(5x^4 - 3ux^2 + 2sx - t)(dt \wedge dx + dy \wedge dz) + 2y(dt \wedge dy - dx \wedge dz) \\
\text{(B.3)} \quad &\quad - 2z(dt \wedge dz + dx \wedge dy) + \varepsilon[(-10x^3 + 3ux - s)(4zdx \wedge dy + 2ydx \wedge dz)]
\end{aligned}$$

Let $W(x, t) = 5x^4 - 3ux^2 + 2sx - t$. Evaluating $\omega(v_1, v_2)$, we obtain:

$$\begin{aligned}
\omega(v_1, v_2) &= \tilde{\omega}(v_1, v_2) = \\
&= \frac{1}{W(x, t)} (\varepsilon(5x^4 - 3ux^2 + 2sx - t)^2 + 4y^2 + 4z^2 + \varepsilon(-10x^3 - 3ux - s)(4y^2 + 4z^2)) \\
&= \frac{1}{W(x, t)} [\varepsilon(5x^4 - 3ux^2 + 2sx - t)^2 + 4y^2(1 + \varepsilon(-10x^3 - 3ux - s)) \\
\text{(B.4)} \quad &\quad + 4z^2(1 + \varepsilon(-10x^3 - 3ux - s))]
\end{aligned}$$

All terms are always positive except possibly $\varepsilon(-10x^3 - 3ux - s)$. By restricting u and s around 0 to a sufficiently small neighbourhood with $|x| < 1$, and by choosing a sufficiently small ε , we can bound this term so that it is smaller than 1, so that the whole expression remains positive.

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